LEARNING ABOUT TOXICITY: WHY ORDER IMBALANCE CAN DESTABILIZE MARKETS

NIHAD ALIYEV, XUE-ZHONG HE, TALIS J. PUTNINS

University of Technology Sydney

ABSTRACT. How does a market digest order imbalance? We show that when market participants learn about the level of adverse selection (the risk of trading against better-informed counterparties) from order flow, a large order imbalance can be destabilizing, causing sharp price movements and evaporation of liquidity, as it signals high toxicity. While such effect is consistent with the practitioner view that order flow is informative about toxicity, it contrasts with standard microstructure models in which the level of adverse selection is assumed to be known and thus order imbalance improves liquidity by revealing private information. Our model helps to understand when markets are most susceptible to imbalance-induced instability and the dynamic process of how markets digest order imbalance.

JEL Classification: G14, D81, D83.

Keywords: Flow toxicity, market instability, order imbalance, multidimensional uncertainty.

Current version: November, 2018.

We thank David Easley, Thierry Foucault, Maureen O'Hara, as well as the conference/seminar participants of 2018 FMA Annual Meeting (San Diego), 2018 Sydney Market Microstructure Meeting, 2018 International Conference on Computing in Economics and Finance (Milan), 2018 Annual Conference of the Multinational Finance Society (Budapest), University of Sydney, and University of Technology Sydney for helpful comments and suggestions. Nihad Aliyev acknowledges financial support from the UTS Business School. Xue-Zhong He acknowledges financial support from the Australian Research Council (DP130103210). Talis Putnins acknowledges financial support from the Australian Research Council (DE150101889).

Corresponding author: Nihad Aliyev, Finance Discipline Group, UTS Business School, University of Technology Sydney, PO Box 123 Broadway, NSW 2007, Australia. Email: Nihad.Aliyev@uts.edu.au.

1. Introduction

Toxic order flow is a source of financial market instability meaning it can cause evaporation of liquidity, elevated volatility, and sharp price movements. Order flow is regarded as toxic when it originates from a better-informed counterparty, causing adverse selection of market participants' orders and losses for liquidity providers. Market practitioners, in particular market makers, have long used order imbalance as an indication of order flow toxicity, adjusting their trading strategies accordingly. Liquidity providers (e.g., algorithmic market makers) often withdraw their quotes in the face of large order imbalances, making markets less liquid during and following large order imbalances (e.g., Chordia, Roll and Subrahmanyam (2002), Anand and Venkataraman (2016)). In the extreme, order imbalances can trigger 'flash crashes'— episodes of extreme price movements accompanied by evaporation of liquidity and elevated volatility (e.g., Easley, López de Prado and O'Hara (2012), Kirilenko et al. (2017)). Given the fundamental importance of market stability in promoting economic growth, it is surprising that we know little about *why* order imbalance can destabilize markets and *when* markets are most vulnerable to destabilizing order imbalance. This paper addresses both of these questions.

Paradoxically, standard market microstructure models with asymmetric information predict that order imbalances stabilize markets ex-post, increasing liquidity and reducing volatility (e.g., Kyle (1985), Glosten and Milgrom (1985)). This prediction follows from the standard assumption that market participants are fully aware of the level of adverse selection (the probability of informed trading and/or the quality of informed traders' information). Under such an assumption, the effect of order imbalance is trivial—it reveals private information about the fundamental value, reducing uncertainty, and thereby increasing liquidity (lower price impacts in the Kyle framework and narrower bid-ask spreads in the Glosten-Milgrom framework). This prediction of standard microstructure models—that we should expect calmer and more liquid markets following periods of large order imbalances is at odds with practice. What is missing from the standard models, we propose, is learning about adverse selection.

Our contribution to the literature is to model the process by which market participants learn about adverse selection risk ('toxicity') from order flow, in particular order imbalance, and study the implications of this learning process. To an otherwise standard sequential trade model, we add uncertainty about the proportion of informed traders (composition uncertainty) and/or the quality of their signals (signal quality uncertainty), resulting in uncertainty about the level of adverse selection. Reflecting a practical challenge faced by real-world liquidity providers, market participants in our model must learn about toxicity, rather than knowing the probability of informed trading and the quality of informed traders' information. This learning occurs from order flow. Intuitively, because informed trading tends to result in order imbalance (informed traders all tend to buy when prices are too low and sell when prices are too high), observing an episode of highly unbalanced order flow acts as a signal that there is likely to be a high proportion of informed traders or that informed traders have very precise information. This upward revision in perceived adverse selection risk can cause liquidity providers to set wider spreads to protect themselves from higher toxicity, as well as sharp price adjustments as the information contained in past order flow is reassessed. Such effects, which all follow from learning about adverse selection, oppose the standard stabilizing effect of order imbalance (learning about fundamental value). The tension between these stabilizing and destabilizing effects is what allows our model to illustrate why order imbalance can sometimes be destabilizing and offer insights about when the destabilizing effects are likely to dominate the stabilizing effects.

We use our model to explore how markets respond to three general order flow patterns — balanced orders, sequences, and reversals. Balanced orders occur when the market maker receives an equal number of buy and sell orders. Sequences are consecutive buy or sell orders. Reversals occur when a sell order follows consecutive buy orders or vice versa. Our analysis delivers four important implications for the dynamics of security prices.

First, balanced orders always stabilize the market. By receiving balanced orders, the market maker maintains her initial beliefs about the security value and revises her belief about adverse selection risk downward. This leads the information content of buy and sell orders to be time-varying and symmetric—informativeness of orders and bid-ask spreads decrease after a period of balanced orders due to lower perceived adverse selection risk. Even this basic effect is in contrast to standard microstructure models with only fundamental value uncertainty because in such models balanced order flow reveals no new information and thus has no effect on prices or liquidity.

Second, a sequence of unbalanced order flow (a series of buys or a series of sells) has two effects, with opposing impacts on liquidity. Unbalanced order flow allows the market maker to learn about the fundamental value (revising beliefs upward in response to buys and downward in response to sells), similar to standard models. This effect tends to make the market more liquid due to reduced uncertainty about the security value. Yet it also leads the market maker to revise her belief about the level of adverse selection risk upward, which tends to make the market less liquid. This means that, unlike in the standard models, order imbalances can be destabilizing. We characterize the necessary and sufficient conditions for order imbalance to be destabilizing. We show that order imbalance destabilizes the market when the initial belief about the adverse selection risk is sufficiently low. This means that financial markets are more vulnerable to order imbalances in times of low perceived toxicity, but can digest more imbalance when toxicity is believed to be high. While this result might seem surprising at first, the intuition is that a large order imbalance when it is not expected presents a larger shock than when the market expects unbalanced order flow.

Third, reversals in order flow (e.g., a sell following a string of buys) can restore liquidity. While this result is intuitive because reversals alleviate the imbalance in order flow received by liquidity providers, it in fact contrasts with standard models and highlights the important role played by learning about adverse selection. In a standard model without learning about adverse selection, a reversal in order flow makes the market less liquid, as it increases uncertainty about the fundamental value. While this effect is also present in our model, an additional effect emerges from learning about adverse selection—a reversal leads the market maker to revise downward her belief about the level of adverse selection risk, which tends to improve liquidity.

A fourth interesting effect of learning about adverse selection, which we term "repricing history", explains accelerating price impacts, asymmetry in the information content of orders, and sharp price movements. When the market maker is uncertain about the proportion of informed traders or the quality of their information, an order has two components to how it impacts the market maker's beliefs about the fundamental value. The first is simply that buys increase the likelihood that the fundamental value is high and vice versa because informed traders tend to buy when the price is below the fundamental value—an effect that drives price discovery in standard models. But a second effect is that the market maker also updates her beliefs about the level of adverse selection or informativeness of order flow and then uses this new belief to reassess what she had learned from past order flow ("repricing history"). If an order increases the market maker's beliefs about the informativeness of order flow, she gives more credit to past orders and prices adjust accordingly (they move in the direction of the imbalance). For example, a market maker that receives a buy after a series of buys will revise upward her beliefs about the informativeness of order flow (due to a larger imbalance), leading her to reassess the past buy imbalance as more informed. Viewing the past buy imbalance as more informed leads to an additional upward revision in the expected fundamental value and thus a larger price increase than in the absence of learning about adverse selection. In fact, this mechanism can lead to accelerating price impacts in trade sequences, similar to those observed empirically during flash crashes. For instance, in a sequence of sells, each subsequent sell not only signals the fundamental value is likely to be low but also signals that the previous sells were more informed than initially believed compounding the downward revision in beliefs about fundamental values.

In addition to accelerating price impacts with continuations in order flow, repricing history also implies the information content of buys and sells will be asymmetric and time-varying, depending on the past order flow. More precisely, reversals in the flow (e.g., a buy after a series of sells, or a sell after a series of buys) decrease the market maker's beliefs about the informativeness of order flow and are more informative than continuations in the flow (buys following buys, or sells following sells). Intuitively, if an order decreases the market makers beliefs about the informativeness of order flow, she gives less credit to past orders and prices adjust accordingly (they move opposite to the direction of the imbalance). Rather than accelerating price impacts, this scenario can result in sharp price reversals. For example, a market maker that receives a sell after a series of buys revises downward her beliefs about the informativeness of order flow (due to a smaller imbalance), leading her to reassess the past buy imbalance as being less informative than previously believed. This leads to an additional downward revision in the likelihood of high fundamental value and a larger price decrease than in the absence of repricing history.

The repricing history effect predicts that the price adjustments to order flow can be particularly sharp due to accelerating price impacts and more informative reversals in order flow. For example, a long string of sell orders similar to flash crashes will lead to accelerating price impact on the way down (high probability of informed trading due to the strong order imbalance), leading to sharp decline in the price. A few buy orders at such time will recover the price quickly due to the repricing history effect. The result is a sharp downward price movement and a quick recovery, amplified by learning about adverse selection.

By accounting for learning about adverse selection, our model provides a rich characterization of the dynamics of security prices in response to order flow and provides intuition about the prevalence of flash crashes with the rise of algorithmic trading. The model explains why price impacts can be asymmetric and time-varying (as has been empirically documented), without turning to frictions such as short selling constraints.[1] The results are also consistent with the empirical research by Hasbrouck (1991) that the trades that arrive when the spread is wide have a greater price impact. The analysis points out that the prevalence of the flash crashes in the algorithmic era may be related to the increased composition and signal quality uncertainty due to the increased complexity of financial markets and their participants.

One way to view the relation between our model and early market microstructure models (e.g., Glosten and Milgrom (1985), Kyle (1985)) is that by adding learning about adverse selection, we allow the model to better reflect the current market structure. At the time of the original models, designated market makers (DMMs) with affirmative obligations to provide two-sided quotes and maintain orderly markets were integral to the functioning of US equity markets. The relative lack of competition faced by DMMs at the time meant they could cross-subsidize liquidity through time, which helped maintain orderly markets and reduce fluctuations in liquidity. They could keep the spread relatively stable, making excess profits in good times (when adverse selection is low) and using those excess profits to subsidize liquidity provision in bad times (when adverse selection is high). Thus, there was less incentive to learn about time-varying adverse selection risk and ensure spreads always reflected the level of toxicity. The abolishment of DMM monopolies and resulting competition in liquidity provision eliminated the ability to cross-subsidize liquidity through time. This is because a liquidity provider without affirmative obligations could undercut the DMMs quotes during good times when adverse selection is low to capture some of the excess profit and step away when adverse selection becomes high. Importantly, efficient learning about the time-varying level of adverse selection, or the 'toxicity' of order flow, allowing liquidity to be priced accurately at every point in time is crucial for a liquidity provider to remain competitive in today's major equity markets. Thus, we argue that our model better reflects the behavior of today's liquidity providers and therefore provides a better description of the dynamics of order flow, liquidity, and prices.

The next section relates this paper to the literature. In Section 3, we introduce a benchmark model that does not require learning about adverse selection to illustrate how order imbalance

[1] Empirical research in market microstructure finds that markets react to buy and sell orders asymmetrically (e.g., Kraus and Stoll (1972), Keim and Madhavan (1996), Chiyachantana et al. (2017)). Saar (2001) characterizes the conditions for the positive and negative price impact asymmetry between buy and sell orders by focusing on short-selling and diversification constraints of the institutional traders.

stabilizes the market by reducing uncertainty about the fundamental value. In Section 4, we extend the model to include uncertainty about adverse selection (proportion of informed traders). In Section 5, we investigate the liquidity and price dynamics in the extended model, and characterize the conditions for liquidity deteriorations and sharp price movements. Section 6 examines the implications of our results for empirical research. Section 7 discusses some extensions and generalizations of our model. Section 8 concludes. The details of extensions and proofs are collected in the appendices.

2. RELATED LITERATURE

Order imbalance can be caused by many factors (e.g., informed trade, macroeconomic variables, "fat-finger" trades). By focusing on traders' demand functions, much of the market crash literature focuses on the causes of order imbalance (e.g., Gennotte and Leland (1990), Barlevy and Veronesi (2003), Hong and Stein (2003)). In this paper, instead of the causes of order imbalance, we focus on its effects. Our model builds on Glosten and Milgrom (1985), which models financial markets as a sequential trading process with one source of uncertainty—the security payoff. Our paper is related to a subset of market microstructure literature that studies environments where market participants face multiple dimensions of uncertainty.

In environments with uncertain information quality, Romer (1993) suggests a possible rational explanation for the October 1987 crash and Blume, Easley and O'Hara (1994) investigate the informational role of volume for technical analysis in a rational expectations framework. In recent studies, Gao, Song and Wang (2013) generate multiple non-linear equilibria with strategic information complementarity and Banerjee and Green (2015) establish empirically relevant return dynamics such as asymmetric reaction to news, volatility clustering, and leverage effects in a rational expectations framework with an uncertain proportion of informed traders. The trading process in a rational expectations framework is not flexible enough to investigate our effects of interest. One reason is that the aggregation of orders in a batch-clearing system prevents them from taking different levels of informativeness at different times. The second and related reason is that in a batch-clearing system trades clear at a single price. Our focus is on how the market maker learns from order flow and when this learning stabilizes and destabilizes financial markets. Therefore, in our model, the dynamics of the quotes and the bid-ask spread play important roles in evaluating the evolution of liquidity.

In a sequential trade model, Easley and O'Hara (1992) introduce "event uncertainty" (uncertainty about whether an event that gives rise to private information about the security value has occurred) to show the relevance of time and volume in the market maker's learning process. Without an information event, the market is only populated by uninformed traders, who (unlike informed traders) sometimes choose not to trade. In this setting, the rate of trade arrivals (trades per unit time) is higher following an information event and therefore the market maker learns from the time between trades. In our model, the market maker always faces an adverse selection problem but to an uncertain degree. Our focus is learning from the order imbalance rather than volume or the pace of trading. Order imbalance is important in how participants would learn about the presence of informed traders and the quality of their information. For example, consider an increase in the arrival intensity of uninformed traders that will increase volume per unit time but not adverse selection. In contrast, an increase in the imbalance between buyers and sellers signals high adverse selection risk and toxicity in the order flow. Our focus on learning from order imbalance rather than the time between trades produces a vastly different set of insights and empirical implications about the dynamics of prices and liquidity. Learning about adverse selection from order imbalance as in our model rather than from the clock time between trades as in Easley and O'Hara (1992) is consistent with recent empirical measures of toxicity, such as VPIN (e.g., Easley, López de Prado and O'Hara (2011)). VPIN seeks to measure toxicity (adverse selection risk) using a volume clock (thereby explicitly disregarding the clock time between trades) based on order imbalances much like how liquidity providers in our model infer the level of adverse selection.[2] Thus, a further contribution of our paper is in providing a theoretical justification for recent empirical toxicity measures such as VPIN.

Avery and Zemsky (1998) propose multiple dimensions of uncertainty with non-monotone signals as a possible explanation for the herd behavior and market mispricing. In our model, we stick to more common monotone information structures that rule out herding and show that order imbalance can destabilize markets when there is an additional source of uncertainty.[3]

[2] Our model (unlike Easley and O'Hara (1992)) is in "event time" or uses a "volume clock" (trade arrivals index time).

[3] The non-monotone signals in Avery and Zemsky (1998) exploit a second source of uncertainty (about whether an information event has occurred or about the precision of informed traders' signals). They assume that if an information event has not occurred, the informed traders know with certainty. However, if an information event has occurred, the informed traders know that an information event has occurred, but they only have a noisy signal about whether it was good or bad news. For this reason, when an informed trader arrives and an information event has occurred, if there has been a significant price run-up, the informed might infer that it is more likely that there has been good news, not bad news even if he receives the noisy bad news signal, and thus he throws away his information and herds.

The destabilizing effects of order imbalance that we analyze are quite different from those in Avery and Zemsky (1998). First, the mechanism is different. In our model, order imbalance reveals information about adverse selection, whereas in Avery and Zemsky order imbalance can be destabilizing because herding can occur and the market maker cannot distinguish between herding and trading on private signals. Second, the nature of the instability is different. In our model, the second source of uncertainty causes order imbalance to move prices more sharply, widen spreads, and increase volatility. Gervais (1997) also studies a sequential trade model in which the market maker is uncertain about the quality of informed traders' signal to argue that financial markets do not necessarily evolve in the direction of efficient markets. In his setting, the evolution of beliefs are path-dependent due to the independence of uncertainties and the bid-ask spread can stick forever at a certain level in which the same equilibrium is repeated in every subsequent period leading to information cascade.

This is the first paper, to our knowledge, to show how learning about the level of adverse selection from the order flow can lead to sudden liquidity dry-ups and sharp price movements in the face of large order imbalances.

3. The Benchmark Model

This section presents a benchmark model that mirrors the classic market microstructure models with uncertainty only about the fundamental value. In this setting, we illustrate the stabilizing effect of order imbalance. The benchmark model allows us to provide a contrast to the subsequent models with composition uncertainty in Section 4 and other sources of uncertainty (i.e., fundamental value, composition, and signal quality uncertainty) in Appendix A.

3.1. **Setup.** We adopt a Glosten-Milgrom framework of one risky security and three types of traders; informed traders, uninformed traders, and a competitive market maker. Trade takes place in $t = 1, ..., T$ periods and the risky security pays off in period $T + 1$. The payoff $\hat{V}$ takes one of two values from the set $\hat{V} \in \{0, 1\}$ with an initial prior probability $\Pr(\hat{V} = 1) = p_1$, where $0 < p_1 < 1$. For ease of exposition, we assume $p_1 = 0.5$ in our analysis and address $p_1 \neq 0.5$ if relevant. Let D_t denote the trade direction, $D_t = -1$ for a sell, $D_t = +1$ for a buy, and P_t denote the transaction price at time t. Public information at time t consists of the sequence of past buys and sells and their transaction prices, denote by $h_t = \{D_\tau, P_\tau\}_{\tau=1}^{t-1}$.[4]

[4] For convenience, unions $\{\cdot, \cdot\}_{\tau=1}^{0}$ are taken to equal $\varnothing$, and both sums $\sum_{\tau=1}^{0}$ and products $\prod_{\tau=1}^{0}$ are taken zero.

As in Glosten-Milgrom type models, the risk-neutral, competitive market maker posts bid and ask quotes, for a fixed volume (normalized to one unit), to earn zero expected profit. At each time t, a trader arrives at the market and can buy at the ask or sell at the bid. With a probability of α the trader arriving at the market is informed and with a probability of $1 - \alpha$ he is uninformed. We focus on interior probability or intensity of informed trading, $\alpha \in (0, 1)$, as this is the empirically relevant case. After each trade, the competitive market maker updates her beliefs about the security payoff and posts new quotes before the next trader arrives.

The informed traders are risk neutral and maximize their expected profits by trading on a serially received signal $\{\theta_t\}$ about the risky security payoff. The signal takes either H (high) or L (low), $\theta_t \in \{H, L\}$, and the quality of the signal is measured by

$$q = \text{Pr}\{\theta_t = H|\hat{V} = 1\} = \text{Pr}\{\theta_t = L|\hat{V} = 0\}, \tag{1}$$

with $q \in (1/2, 1]$. When $q = 1$, the informed traders' information is perfect. When $q = 1/2$, the signal is completely uninformative. By Bayes' theorem, an informed trader who receives $\theta_t = H$ will revise his private value to

$$v_t^H = \frac{p_t \cdot q}{[p_t \cdot q + (1 - p_t) \cdot (1 - q)]} > p_t, \tag{2}$$

and who receives $\theta_t = L$ will revise his private value to

$$v_t^L = \frac{p_t \cdot (1 - q)}{[p_t \cdot (1 - q) + (1 - p_t) \cdot q]} < p_t, \tag{3}$$

where $p_t = \text{Pr}(\hat{V} = 1|h_t) = E_t[\hat{V}|h_t]$ is the current expected value of $\hat{V}$ conditional on the public information history h_t.

The uninformed traders trade according to their liquidity needs or hedging purposes, which are exogenous to the model. For convenience, we assume that they buy and sell with equal probabilities with perfectly inelastic demand.[5] The structure of the economy described so far is common knowledge among all participants.

3.2. **Equilibrium.** The standard Bertrand competition argument that the competitive market maker expects a zero profit implies that the market maker's bid (ask) quote is the expected future payoff of the risky security conditional on receiving a sell (buy) order. That is, the bid

[5] In Section 7, we discuss the impacts of allowing discretionary uninformed trading on our results.

price $B_t = E_t[\hat{V}|h_t, D_t = -1]$ and the ask price $A_t = E_t[\hat{V}|h_t, D_t = +1]$. Now we formally define the equilibrium for the benchmark economy.

Definition 1. *An equilibrium consists of the market maker's prices, informed traders' trading strategies, and posterior beliefs such that:*

(i) the bid and ask prices satisfy the zero-expected-profit condition, given the market maker's posterior beliefs;

(ii) the informed traders at time t maximize their expected profits given the signal θ_t and the public information history h_t;

(iii) the beliefs satisfy Bayesian updating.

In the benchmark equilibrium with only fundamental value uncertainty, an informed trader who arrives at the market with $\theta_t = H$ ($\theta_t = L$) will buy (sell) if his private valuation is higher (lower) than the ask (bid) price at time t, $v_t^H > A_t$ ($v_t^L < B_t$). In equilibrium, $v_t^H > A_t$ and $v_t^L < B_t$, and therefore the informed traders always trade in the direction of their information.[6] This characterizes the equilibrium A_t and B_t in the following proposition.

Proposition 2. *The equilibrium bid and ask prices are respectively given by:*

$$B_t = \frac{p_t}{p_t + \delta \cdot (1 - p_t)}, \tag{4}$$

$$A_t = \frac{p_t}{p_t + \delta^{-1} \cdot (1 - p_t)}, \tag{5}$$

and the bid-ask spread is given by

$$S_t = \frac{p_t \cdot (1 - p_t) \cdot (\delta - \delta^{-1})}{\big(p_t + \delta \cdot (1 - p_t)\big) \cdot \big(p_t + \delta^{-1} \cdot (1 - p_t)\big)}, \tag{6}$$

where $p_t = E_t[\hat{V}|h_t]$ and $\delta = \frac{1+\alpha\cdot(2\cdot q-1)}{1-\alpha\cdot(2\cdot q-1)}$. In addition, δ is always greater than unity and increases with the intensity of informed trading α and the quality of the informed traders' private information q.

In this equilibrium, the market is always open. This is because the market maker can always set a spread wide enough to recoup from the uninformed traders the losses she expects to incur

[6] The reason is that if B_t and A_t are set less than v_t^L and higher than v_t^H (i.e., $B_t < v_t^L$ and $A_t > v_t^H$), then no informed traders would trade and all trades would arise from the uninformed traders. The competitive, zero expected profit B_t and A_t without any informed trading are equal to the current expected security value, $B_t = A_t = p_t$, with zero spread. Because the signals are always informative ($q > 1/2$), $v_t^H > p_t$ and $v_t^L < p_t$ and therefore in a competitive equilibrium, B_t and A_t cannot be less than v_t^L and higher than v_t^H respectively.

from the informed traders. In addition, the market maker sets a wider spread with the intensity of informed trading and the quality of their signals. It follows from Eqs. (4) and (5) that the bid price decreases and the ask price increases with the informativeness of orders δ, leading to a wider bid-ask spread. Since δ increases with the intensity of informed trading and the quality of their signals, it measures the informativeness of orders and the adverse selection risk.

3.3. **The dynamics of the quotes and the bid-ask spread.** In this paper, we are particularly interested in the dynamics of the quotes and the bid-ask spread. For this purpose, we characterize the dynamics of the risky payoff as

$$p_{t+1} = E_{t+1}[\hat{V}|h_t, D_t, P_t] = E_{t+1}[\hat{V}|h_{t+1}] = \begin{cases} B_t, & \text{if} \quad D_t = -1, \\ A_t, & \text{if} \quad D_t = +1, \end{cases} \tag{7}$$

and re-express in a particularly convenient form in the following lemma. Eq. (7) follows from the fact that, in this setting, the current expected security value is the last realized transaction price.

Lemma 3. *Let $N_t = \sum_{\tau=1}^{\tau=t-1} D_\tau$ be the order imbalance up to (but not including) the trade at time t (number of buys minus number of sells). Then the dynamic expectations of the market maker about the risky security payoff satisfy*

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot \delta^{D_t}, \tag{8}$$

and hence

$$p_t = \frac{\delta^{N_t}}{1+\delta^{N_t}}. \tag{9}$$

Eq. (8) shows that the odds of a high future value are revised upward following a buy and downward following a sell. The amount by which the expectations are revised is determined by the informativeness of trades. More precisely, the revision in the expectation about the payoff is stronger with more informative trades (or more informative trades have higher price impacts). Eq. (9) shows that all of the information contained in the past trades and prices can be represented by the order imbalance, N_t, a sufficient statistic for the history of the order flows. This means that the trade sequences that do not change the order imbalance (i.e., balanced order flows) do not change the market maker's beliefs about the security payoff. Thus, the expected payoff and the bid-ask spread at any point in time can be expressed succinctly as a function of the order imbalance up to that point in time and the informativeness of trades.

To further facilitate interpretation, we insert Eq. (9) into Eq. (6) and re-express the bid-ask spread as a function of the informativeness of orders and order imbalance as

$$S_t = \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})}. \tag{10}$$

Interestingly, Eq. (10) also shows that the bid-ask spread decreases with order imbalances in either direction, excess buy or sell orders, and takes its maximum value of

$$\bar{S} = \frac{\delta - 1}{\delta + 1}, \tag{11}$$

with balanced order flow (i.e., $N_t = 0$).[7]

In Figure 1, we illustrate the uncertainty about the payoff, p_t, and the bid-ask spread in the face of large order imbalances for three possible values of the informativeness of orders. Panel (a) illustrates that the market maker revises the expected value of the security payoff upward when she has a positive order imbalance and downward when she has a negative order imbalance. Moreover, the upward and downward revisions are larger with more informative trades. However, irrespective of the informativeness of orders, uncertainty about the payoff is highest when the market maker has balanced orders. Panel (b) shows that the spread is maximum when the market maker has balanced orders and declines in response to order imbalances in either direction. This illustrates the stabilizing role of order imbalance in the benchmark model with only fundamental value uncertainty. Intuitively, order imbalance is informative about the risky payoff and therefore resolves uncertainty (either $p_t \to 0$ or $p_t \to 1$). Moreover, the bid-ask spread declines faster with more informative trades because uncertainty is resolved faster. Formally, we have the following corollary.

Corollary 4. *In the presence of uncertainty only about the security payoff;*

(i) the market maker observing balanced order flows (i.e., $N_t = 0$) learns nothing and therefore does not update her beliefs about the payoff;

[7] In general, the maximum spread occurs when the market maker has maximum uncertainty (i.e., $p_t = 0.5$) about the payoff. With a balanced order flow, the market maker learns nothing and sustains her initial maximum uncertainty (i.e., $p_1 = 0.5$). In fact, when $p_1 > 0.5$ (resp. $p_1 < 0.5$), the same maximum spread corresponds to a negative (resp. positive) order imbalance. The reason for this is that the maximum uncertainty, $p_t = 0.5$, occurs with a negative (resp. positive) order imbalance when $p_1 > 0.5$ (resp. $p_1 < 0.5$).

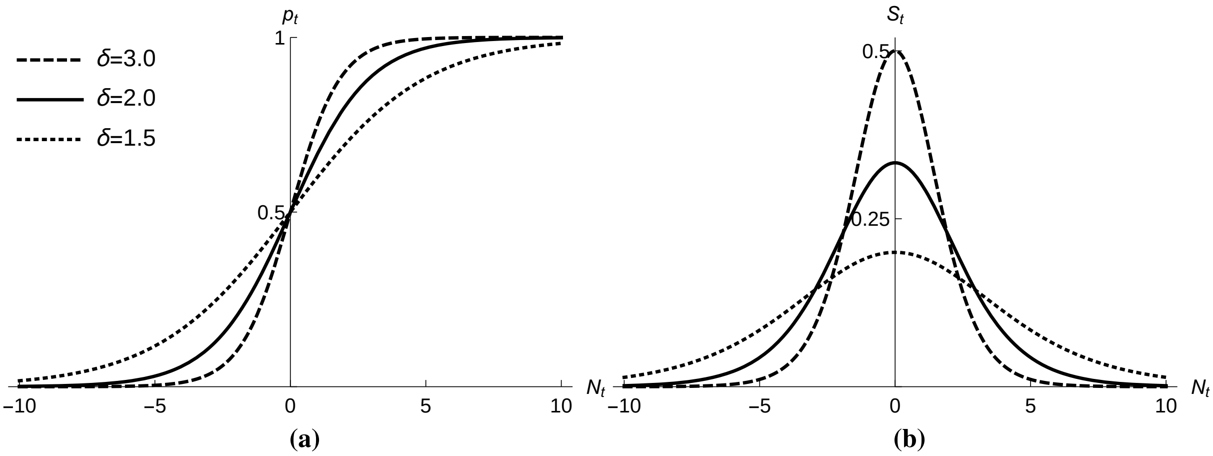

Figure 1. The dynamics of the belief p_t and the spread S_t with respect to order imbalance N_t. Panel (a) plots the conditional expected value of the payoff p_t and (b) plots the spread S_t against the order imbalance for three different values of informativeness of trades, $\delta = 1.5$, 2, and 3 when $p_1 = 0.5$.

(ii) the market maker with a positive (negative) order imbalance increases (decreases) the conditional expected value of the payoff and the magnitude of the increase (decrease) is larger with more informative trades;

(iii) with balanced order flows (i.e., $N_t = 0$), the bid-ask spread S_t at time t equals the initial bid-ask spread S_1;

(iv) the bid-ask spread narrows with order imbalances in either direction and converges to zero as the order imbalance goes to infinity (i.e., order imbalance stabilizes the market).

The stabilizing role of order imbalances in the benchmark model hinges upon having only uncertainty about the fundamental value of the security. These results are at odds with what we observe in financial markets during and following large order imbalances. The experience of the U.S. financial markets on May 6, 2010, ("Flash Crash") and treasury markets on October 15, 2014, ("Flash Rally") are recent extreme examples of the destabilizing role of negative and positive order imbalances, respectively. Similar results are also observed during the global financial crisis in 2007-2009, Asian financial crisis in 1997-1998, October 1987 crash, and many other extreme events (e.g., Easley and O'Hara (2010), Scholes (2000)). The destabilizing role of order imbalance is not confined to the aggregate market level extreme events. On a smaller scale, instantaneous price moves due to the destabilizing role of order imbalance are more common with the rise of algorithmic trading. In practice, order imbalance is an indication of the toxicity in order flow. Unlike the practice, however, in the benchmark model, order imbalance merely serves to convey information about the fundamental value of the security.

The bid-ask spread arises entirely due to the known adverse selection risk of the competitive market maker and approaches to zero in the face of large order imbalances — order imbalance stabilizes the market when the market maker knows the true information structure of the market.

4. Learning about adverse selection

In this section, we introduce an additional source of uncertainty about the adverse selection of the market maker to explore a destabilizing role of order imbalance. Adverse selection risk is a function of the number of informed traders and the quality of their information. Adding uncertainty and learning about either of these parameters produces qualitatively similar results and therefore in the interests of simplicity, we focus on uncertainty about the number of informed traders.

Two key differences distinguish this model from the benchmark model. First, the market maker's quotes are affected not only by beliefs about the security payoff, parameters affecting the adverse selection risk (i.e., the probability of informed trading and the quality of informed traders' information), but also uncertainty about the adverse selection risk. Second, the market maker's beliefs about adverse selection risk change over time as the trading process evolves.

4.1. **Uncertainty about the proportion of informed traders.** We keep all the features of the benchmark model as described in Section 3 (we set $q = 1$ for notational simplicity) and incorporate uncertainty about the composition of market participants. We assume that the probability of informed trading takes either low or high values from the set $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ with an initial prior probability of $\Pr(\hat{\alpha} = \alpha_H) = \pi_1$, where $0 < \alpha_L < \alpha_H < 1$ and $0 < \pi_1 < 1$. Without perfect knowledge about the fractions of the informed and uninformed traders in the market, the market maker's beliefs about the composition of traders change depending on the evolution of the trading process.[8] In what follows, we denote the market maker's belief about the high proportion of informed traders in the market conditional on the trading history as $\pi_t = \Pr\{\hat{\alpha} = \alpha_H | h_t\}$. With two possible values for the probability of informed trading and the risky payoff (i.e., $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ and $\hat{V} \in \{0, 1\}$), there are four different combinations of the level informed trading and the payoff realization. Denote the states $S \in \{s_1, s_2, s_3, s_4\}$, where

[8] Making other market participants (in addition to the market maker) uncertain about the composition of market participants does not affect the model since the informed traders do not use the price function to extract information about the eventual security payoff and the uninformed traders are assumed to trade exogenously. We discuss a generalization of this assumption in Section 7.

$$s_1 = \{\hat{\alpha} = \alpha_H, \hat{V} = 1\}, \qquad s_2 = \{\hat{\alpha} = \alpha_H, \hat{V} = 0\},$$
$$s_3 = \{\hat{\alpha} = \alpha_L, \hat{V} = 1\}, \qquad s_4 = \{\hat{\alpha} = \alpha_L, \hat{V} = 0\}. \tag{12}$$

Since these states are disjoint the market maker's beliefs about the payoff and the composition of traders at time t are, respectively, given by

$$p_t = \text{Pr}(\hat{V} = 1|h_t) = \text{Pr}(s_1|h_t) + \text{Pr}(s_3|h_t), \tag{13}$$

$$\pi_t = \text{Pr}(\hat{\alpha} = \alpha_H|h_t) = \text{Pr}(s_1|h_t) + \text{Pr}(s_2|h_t). \tag{14}$$

It is very intuitive to expect different order flows with different compositions of market participants. Intuitively, a higher proportion of informed traders is more likely to result in larger positive (negative) order imbalances when $\hat{V} = 1$ ($\hat{V} = 0$). The market maker observing more unbalanced orders than expected therefore will increase her belief about the high proportion of informed traders in the market and vice versa. Before considering the market maker's learning problem in detail, we characterize the equilibrium quotes and bid-ask spread in the trading period t, extending the results from the benchmark model.

Proposition 5. *The equilibrium bid and ask prices in the presence of composition uncertainty are respectively given by*

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1 - p_t)}, \tag{15}$$

$$A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)}, \tag{16}$$

and the bid-ask spread is given by

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\big(p_t + \delta_t^s \cdot (1 - p_t)\big) \cdot \big(p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)\big)}, \tag{17}$$

where

$$\delta_t^b = \frac{(1 + \alpha_H) \cdot Pr(s_1|h_t) + (1 + \alpha_L) \cdot Pr(s_3|h_t)}{(1 - \alpha_H) \cdot Pr(s_2|h_t) + (1 - \alpha_L) \cdot Pr(s_4|h_t)} \cdot \left(\frac{1 - p_t}{p_t}\right) \tag{18}$$

and

$$\delta_t^s = \frac{(1 + \alpha_H) \cdot Pr(s_2|h_t) + (1 + \alpha_L) \cdot Pr(s_4|h_t)}{(1 - \alpha_H) \cdot Pr(s_1|h_t) + (1 - \alpha_L) \cdot Pr(s_3|h_t)} \cdot \left(\frac{p_t}{1 - p_t}\right) \tag{19}$$

show the informativeness of buy and sell orders respectively. In addition, δ_t^b and δ_t^s are always greater than unity and increase with the intensities of informed trading α_L and α_H.

The forms of the bid, ask prices and the spread are familiar from the benchmark model (see Eqs. (4), (5), and (6)) and most of the intuitions carry forward. There are three main differences from the benchmark model (see Proposition 2). First, the market maker's belief about the high informed trading π_t in Eq. (14) is time-varying. Second, unlike the benchmark model with constant informativeness of orders for buy and sell orders, the information content of buy and sell orders in Eqs. (18) and (19) are different and vary through time due to the time-varying beliefs about the composition of traders and the security payoff.[9] This leads to asymmetric reactions of the bid and ask quotes in response to buy and sell orders. Third, the dynamics of the expectations about the payoff p_t are not only affected by the order imbalance and the (constant) informativeness of orders as in the benchmark model, but also by the changing beliefs about the composition of market participants and the time-varying informativeness of orders. For example, as the orders become more informative, the market maker learns the fundamental value faster. We formalize this intuition in the following lemma by characterizing the dynamics of the beliefs about the risky payoff.

Lemma 6. *The dynamic expectations of the market maker about the risky security payoff satisfy*

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot \delta_t^b \quad \textit{if} \quad D_t = +1, \tag{21}$$

and

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot (\delta_t^s)^{-1} \quad \textit{if} \quad D_t = -1. \tag{22}$$

Lemma 6 shows how p_{t+1} is obtained from p_t when the market maker observes a buy or a sell order. It is straightforward to check that Lemma 6 reduces to Lemma 3 in the benchmark model when the market maker knows the composition of traders. In fact, when $\alpha = \alpha_L = \alpha_H$,

[9] Unlike the benchmark model, the informativeness of buy and sell orders, δ_t^b and δ_t^s, involve the market maker's belief about the risky payoff along with the parameters of the adverse selection risk. This occurs because over time the market maker's belief about the risky payoff and the composition of traders are dependent. The best way to see that δ_t^b and δ_t^s are indeed analogous to the informativeness of orders δ in the benchmark model is to assume independence between $\hat{\alpha}$ and $\hat{V}$. Then, substituting independence conditions (i.e., $\Pr(s_1|h_t) = \pi_t \cdot p_t$, $\Pr(s_2|h_t) = \pi_t \cdot (1-p_t)$, $\Pr(s_3|h_t) = (1-\pi_t) \cdot p_t$ and $\Pr(s_4|h_t) = (1-\pi_t) \cdot (1-p_t)$) into Eqs. (18) and (19) leads to symmetric informativeness for buys and sells

$$\delta_t^b = \delta_t^s = \frac{1 + (\pi_t \cdot \alpha_H + (1-\pi_t) \cdot \alpha_L)}{1 - (\pi_t \cdot \alpha_H + (1-\pi_t) \cdot \alpha_L)}, \tag{20}$$

which is only dependent on the parameters of adverse selection as in the benchmark model. Additionally, assuming independence about the composition of traders and the fundamental value leads the sequence of orders to matter (path-dependence) in the model, consistent with the empirical findings in Hausman, Lo and MacKinlay (1992). The path-dependence is not integral for our analysis and the sufficient statistic in our setting is order imbalance and time (N_t, t), yet the destabilizing role of order imbalance with a different magnitude is also present with path-dependence.

it follows from Eqs. (18) and (19) that $\delta_t^b = \delta_t^s = \delta = \left(\frac{1+\alpha}{1-\alpha}\right)$ leading to Eq. (8). Similar to Eq. (8) in the benchmark model, Eqs. (21) and (22) show that the odds of a high future value are revised upward following a buy and downward following a sell. Unlike the benchmark model, however, the upward (resp. downward) revisions in δ_t^b and δ_t^s lead to stronger (resp. weaker) revisions in p_t.

4.2. **Learning about the payoff and proportion of informed traders.** In this subsection, we study how the market maker learns about the proportion of informed traders. The task is to determine how π_t is updated when the market maker observes order flow up to time t. One way to understand the learning process of the market maker is to consider the probability of buy and sell orders in different states. A buy order is most likely to occur in a market with a high proportion of informed traders and high security payoff (i.e., $S = s_1$). Similarly, a sell order is most likely to occur in a market with a high proportion of informed traders and low security payoff (i.e., $S = s_2$). More precisely, it follows from the definitions of the states that the probability of an order $D_t \in \{-1, 1\}$ in each state is given by

$$\begin{aligned} \Pr(D_t|s_1) &= \frac{1+\alpha_H \cdot D_t}{2}, \qquad \Pr(D_t|s_2) = \frac{1-\alpha_H \cdot D_t}{2}, \\ \Pr(D_t|s_3) &= \frac{1+\alpha_L \cdot D_t}{2}, \qquad \Pr(D_t|s_4) = \frac{1-\alpha_L \cdot D_t}{2}. \end{aligned} \tag{23}$$

The probabilities in different states imply that in the presence of uncertainty about the proportion of informed traders, the *direction* and *amount* of order imbalance are informative about the payoff and only the *amount* of order imbalance is informative about the proportion of informed traders. The *direction* is informative about the payoff because high fundamental value states (i.e. s_1 and s_3) have higher buy and lower sell probabilities than the corresponding low fundamental value states (i.e., s_2 and s_4), leading the positive (resp. negative) imbalance to increase the probabilities of s_1 and s_3 (resp. s_2 and s_4). The *amount* of imbalance is informative about the payoff since more unbalanced buy (resp. sell) orders increase s_1 more than s_3 (resp. s_2 more than s_4). The *amount* of imbalance is also informative about the proportion of informed traders because buy and sell probabilities in low informed states (i.e., s_3 and s_4) are closer to 0.5, implying that balanced orders will increase low informed states and unbalanced orders will increase high informed states (i.e., s_1 and s_2). However, the *direction* of imbalance is uninformative about the proportion of informed because excess buy orders increase s_1 in the same way as excess sell orders increase s_2, leading to the same belief about the informed trading.

As the market maker is Bayesian, her belief about the particular state given the trading history h_t with b_t buy and s_t sell orders follows from Bayes rule as

$$\Pr(s_1|h_t) = \frac{p_1 \cdot \pi_1 \cdot \left(1+\alpha_H\right)^{b_t}\left(1-\alpha_H\right)^{s_t}}{f(h_t)}, \tag{24}$$

where

$$\begin{aligned} f(h_t) = & p_1 \cdot \pi_1 \cdot (1+\alpha_H)^{b_t}(1-\alpha_H)^{s_t} + (1-p_1) \cdot \pi_1 \cdot (1-\alpha_H)^{b_t}(1+\alpha_H)^{s_t} \\ & + p_1 \cdot (1-\pi_1)(1+\alpha_L)^{b_t}(1-\alpha_L)^{s_t} + (1-p_1)(1-\pi_1)(1-\alpha_L)^{b_t}(1+\alpha_L)^{s_t}. \end{aligned} \tag{25}$$

The probabilities of the other states are calculated similarly. By Eqs. (13) and (14), the revision in each state's probability after the trading history h_t is reflected in the market maker's beliefs about the security payoff and the proportion of informed traders as

$$p_t = \frac{p_1 \cdot \pi_1 \cdot \left(1+\alpha_H\right)^{b_t}\left(1-\alpha_H\right)^{s_t} + p_1 \cdot (1-\pi_1) \cdot \left(1+\alpha_L\right)^{b_t}\left(1-\alpha_L\right)^{s_t}}{f(h_t)}, \tag{26}$$

$$\pi_t = \frac{p_1 \cdot \pi_1 \cdot \left(1+\alpha_H\right)^{b_t}\left(1-\alpha_H\right)^{s_t} + (1-p_1) \cdot \pi_1 \cdot \left(1-\alpha_H\right)^{b_t}\left(1+\alpha_H\right)^{s_t}}{f(h_t)}. \tag{27}$$

Eqs. (26) and (27) show that the order imbalance for the given number of trades (event time) determines the market maker's beliefs about the payoff and proportion of informed traders since $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$. It follows from Eq. (26) that the market maker that receives balanced order flow learns nothing about the fundamental value as in the benchmark model (i.e., $p_t = p_1$). Eq. (26) also show that all else equal, the greater the positive (resp. negative) imbalance, the more likely it is that the market maker believes the fundamental value of the security is high (resp. low). Additionally, it follows from Eq. (27) that all else equal, the greater the imbalance, the more likely it is that the market maker believes the market is highly populated by informed traders. We summarize the effects of order imbalance for the given number of trades on the market maker's beliefs in the following proposition.

Proposition 7. *(i) The market maker's expected value of the security payoff is unchanged with zero order imbalance and increases (resp. decreases) with positive (resp. negative) order imbalance; that is* $p_t = p_1$ *when* $N_t = 0$ *and* $\frac{\partial p_t}{\partial N_t} > 0$. *(ii) the market maker's belief about the high informed trading increases with order imbalance in either direction; that is* $\frac{\partial \pi_t}{\partial |N_t|} > 0$.

To facilitate interpretation, in Figure 2, we contrast the market maker's expected value of the security payoff p_t and belief about the high proportion of informed traders π_t in the benchmark

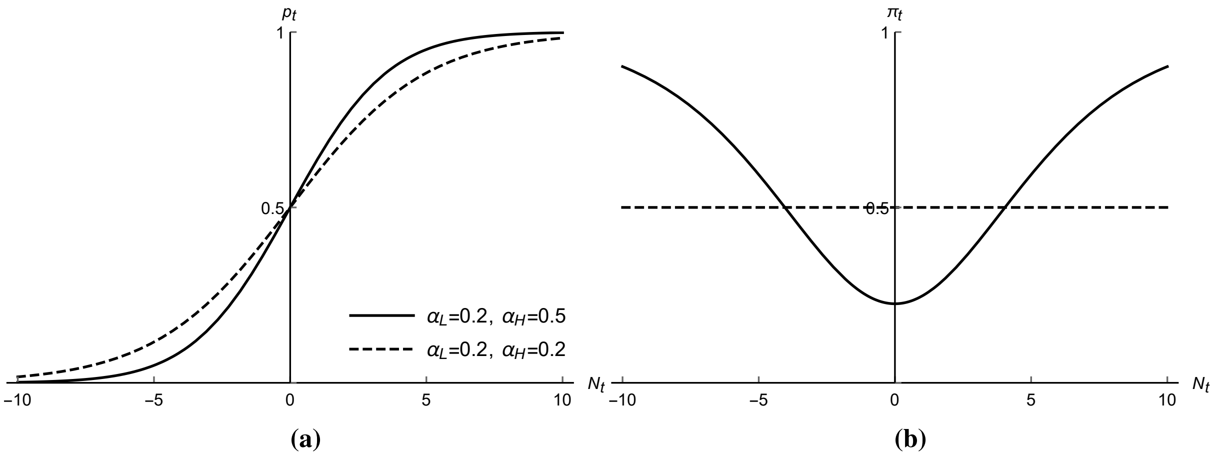

Figure 2. The dynamics of the beliefs p_t and π_t with respect to order imbalance N_t.
Panel (a) plots the conditional expected value of the payoff p_t and (b) plots the belief about the high proportion informed trading π_t against the order imbalance for the given time $t = 11$ for two different values of low and high informed trading $\alpha_L = \alpha_H = 0.2$ (benchmark), and $\alpha_L = 0.2$ and $\alpha_H = 0.5$ (uncertain proportion of informed traders). The initial beliefs are $p_1 = 0.5$ and $\pi_1 = 0.5$.

and extended models. Panel (a) illustrates that, in both models, the market maker revises the expected value of the security payoff upward (resp. downward) when she has a positive (resp. negative) order imbalance. What is different in the presence of uncertainty about the adverse selection is that the upward and downward revisions are larger with more order imbalance. The reason for this effect is illustrated in Panel (b), which shows that the market maker's belief about the high proportion of informed traders π_t increases with order imbalance. The market maker learns about the fundamental value faster with more order imbalance since more order imbalance signals the presence of more informative orders. Additionally, Panel (b) shows that the market maker's beliefs about the high proportion of informed traders π_t is higher (resp. lower) than her initial belief π_1 when the order imbalance for the given time is sufficiently high (resp. low). Focusing on two extreme cases (zero order imbalance and maximum order imbalance for the given time), the following corollary formalizes these observations.

Corollary 8. *(i) The market maker observing balanced order flows (i.e., $N_t = 0$ at time t) revises her belief about the high informed trading downward (i.e., $\pi_t < \pi_1$). (ii) The market maker observing sequences of buy or sell orders (i.e., $N_t = t - 1$ or $N_t = -(t - 1)$ at time t) revises her belief about the high informed trading upward (i.e., $\pi_t > \pi_1$).*

There are two reasons why these results are of interest. First, in the presence of uncertainty about the proportion of informed traders, balanced orders will stabilize the market by reducing the bid-ask spread since the market maker retains her initial belief about the payoff but revises her belief about the high informed trading downward. This is in contrast to the benchmark model, where the market maker observing balanced orders learns nothing and maintains her initial bid-ask spread (see Corollary 4). Second, consecutive buy or sell orders can destabilize financial markets by widening the bid-ask spread since they signal the presence of high informed trading. This is also in contrast to the benchmark model, where the market maker that receives sequences of buy or sell orders only learns about the payoff and narrows the spread due to the resolution of uncertainty about the fundamental value. In the presence of uncertainty about the proportion of informed traders, the market maker during a period of large and temporary selling (resp. buying) pressure such as the Flash Crash (resp. Flash Rally) updates the expected value of security payoff downward (resp. upward). While this resolution of uncertainty about the fundamental value puts downward pressure on the bid-ask spread, there is an opposing effect on the spread from learning about the adverse selection. The market maker also updates her belief about the high informed trading in the market and widens the bid-ask spread due to the increase in the likelihood of high adverse selection risk (high informed trading). Ultimately, whether the spread widens or narrows following periods of large order imbalance depends on which effect dominates.

5. Liquidity and Price dynamics

Since we are interested in the role of learning about toxicity in stabilizing and destabilizing financial markets in the face of different order flow patterns, in this section, we investigate the liquidity and price dynamics in our model.

5.1. **Liquidity distortions.** We first examine the effects of different order flows on the evolution of bid-ask spread to evaluate liquidity in the presence of uncertainty about adverse selection. For this purpose, we consider the amount of deviation of the bid-ask spread at time t from the initial spread.[10] It follows from Proposition 5 that the initial spread is given by

[10] We choose the amount of deviation of the bid-ask spread at time t from the initial spread rather than the deviation from the benchmark spread for two reasons. First, the deviation from the initial spread shows the net liquidity distortion due to learning about the fundamental value and adverse selection, which can further be decomposed into the liquidity distortion due to each learning component, whereas the deviation from the benchmark spread measures the liquidity distortion due to learning only about adverse selection. Second, we are interested in a stricter condition to analyze whether order imbalance is destabilizing. The most strict condition to examine destabilizing order imbalance is to show that the spread is increasing in the order imbalance until a certain threshold order imbalance. The complexity of Eq. (17) after inserting state probabilities makes a full analytical characterization of the partial derivative $\frac{\partial S_{\alpha,t}}{\partial N_t}$ impractical. The second most strict condition is the deviation from the initial spread since the initial spread is higher than the benchmark spread in the face of a large order imbalance.

$$S_1 = \frac{\delta_1 - 1}{\delta_1 + 1} = \pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L, \tag{28}$$

with equal informativeness of buy and sell orders,

$$\delta_1^b = \delta_1^s = \delta_1 = \frac{1 + (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)}{1 - (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)}. \tag{29}$$

Combining the bid-ask spread in the presence of uncertainty about adverse selection at time t in Eq. (17) and the initial spread in Eq. (28) yields

$$\Delta S_t = S_{\alpha,t} - S_1, \tag{30}$$

where ΔS_t given by Eq. (B-48) in Appendix B is the net liquidity distortion at time t relative to the initial spread. The net liquidity distortion ΔS_t implies stabilizing (resp. destabilizing) order flow when $\Delta S_t < 0$ (resp. $\Delta S_t > 0$). In addition, ΔS_t includes both the effects of learning about the fundamental value and proportion of informed traders on the spread. To examine the contributions of each learning component on the net liquidity distortion, we decompose ΔS_t into two components — the distortion due to learning only about the fundamental value and the distortion due to learning about the adverse selection. Formally, the net liquidity distortion is given by

$$\Delta S_t = \Delta S_t^V + \Delta S_t^A, \tag{31}$$

where $\Delta S_t^V = S_t - S_1$ is the liquidity distortion due to learning about the fundamental value and $\Delta S_t^A = S_{\alpha,t} - S_t$ is the liquidity distortion due to learning about the adverse selection. We investigate the net liquidity distortion ΔS_t and its contributors ΔS_t^V and ΔS_t^A during three general order flow patterns: balanced orders, consecutive buy or sell orders, and reversals. As we investigate each order flow pattern, we refer to Figure 3, where we contrast the dynamics of belief about the proportion of informed traders, quotes, and spreads of the market maker that faces this order flow in the benchmark and extended models.

For balanced order flows (i.e., $N_t = 0$ or $b_t = s_t$), the informativeness of buy and sell orders at time t are given by

$$\delta_t^b = \delta_t^s = \frac{\pi_1 \cdot (1 + \alpha_H)^{s_t+1} \cdot (1 - \alpha_H)^{s_t} + (1 - \pi_1) \cdot (1 + \alpha_L)^{s_t+1} \cdot (1 - \alpha_L)^{s_t}}{\pi_1 \cdot (1 + \alpha_H)^{s_t} \cdot (1 - \alpha_H)^{s_t+1} + (1 - \pi_1) \cdot (1 + \alpha_L)^{s_t} \cdot (1 - \alpha_L)^{s_t+1}} < \delta_1, \tag{32}$$

leading to

$$\Delta S_t = \Delta S_t^A = \frac{2 \cdot (\delta_t^s - \delta_1)}{\left(1 + \delta_t^s\right) \cdot \left(1 + \delta_1\right)} < 0, \tag{33}$$

since the market maker in the benchmark model learns nothing and retains her initial spread (i.e., $S_t = S_1$ or $\Delta S_t^V = 0$). Eq. (33) always takes a negative value (i.e., $\Delta S_t < 0$ for $\delta_t^s < \delta_1$), meaning that in the presence of uncertainty about adverse selection, balanced order flow is always stabilizing since it results in the narrower spread relative to the initial (also the benchmark) spread. This is because the market maker that observes balanced order flow retains her initial belief about the security payoff (see Proposition 7), but revises down her belief about the high informed trading (see Corollary 8) as illustrated in Panel (a1) of Figure 3. Panels (a2)-a(3) of Figure 3 illustrate that balanced order flow is stabilizing in the extended model, whereas it has no effect on prices or liquidity in the benchmark model.

For consecutive sell orders (i.e., $N_t = -(t-1)$), the informativeness of buy and sell orders at time t are respectively given by

$$\delta_t^b = \left[\frac{\pi_1 \cdot (1+\alpha_H) \cdot (1-\alpha_H)^{t-1} + (1-\pi_1) \cdot (1+\alpha_L) \cdot (1-\alpha_L)^{t-1}}{\pi_1 \cdot (1-\alpha_H) \cdot (1+\alpha_H)^{t-1} + (1-\pi_1) \cdot (1-\alpha_L) \cdot (1+\alpha_L)^{t-1}}\right] \cdot \left[\frac{p_t}{1-p_t}\right]^{-1}, \tag{34}$$

$$\delta_t^s = \left[\frac{\pi_1 \cdot (1+\alpha_H)^t + (1-\pi_1) \cdot (1+\alpha_L)^t}{\pi_1 \cdot (1-\alpha_H)^t + (1-\pi_1) \cdot (1-\alpha_L)^t}\right] \cdot \left[\frac{p_t}{1-p_t}\right], \tag{35}$$

where

$$\frac{p_t}{1-p_t} = \frac{\pi_1 \cdot (1-\alpha_H)^{t-1} + (1-\pi_1) \cdot (1-\alpha_L)^{t-1}}{\pi_1 \cdot (1+\alpha_H)^{t-1} + (1-\pi_1) \cdot (1+\alpha_L)^{t-1}}. \tag{36}$$

Unlike balanced order flow, Eqs. (34) and (35) show that during unbalanced orders the informativeness of buy and sell orders are asymmetric, impacting the beliefs about the fundamental value asymmetrically. The following corollary characterizes the association between the beliefs about the fundamental value and the informativeness of orders during unbalanced order flow.

Corollary 9. *(i) With a sequence of buy orders (i.e., $N_t = t-1$),*

$$p_t = \frac{\prod_{i=1}^{t-1} \delta_i^b}{1 + \prod_{i=1}^{t-1} \delta_i^b}, \tag{37}$$

(ii) With a sequence of sell orders (i.e., $N_t = -(t-1)$),

$$p_t = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1}}{1 + \prod_{i=1}^{t-1} (\delta_i^s)^{-1}}. \tag{38}$$

Corollary 9 implies that when there is a consecutive buying or selling pressure in the market, the geometric mean of the informativeness of buy or sell orders up to time t (i.e., $(\prod_{i=1}^{t-1} \delta_i^b)^{\frac{1}{t-1}}$ or $(\prod_{i=1}^{t-1} \delta_i^s)^{\frac{1}{t-1}}$) play the same role as the constant informativeness orders in the benchmark model in determining p_t. Inserting the belief dynamics during sell sequences in Eq. (38) into the characterization of the spread in Eq. (17) obtains the spread as

$$S_{\alpha,t} = \frac{\prod_{i=1}^{t-1}(\delta_i^s)^{-1} \cdot \big(\delta_t^s - (\delta_t^b)^{-1}\big)}{\big(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \delta_t^s\big) \cdot \big(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + (\delta_t^b)^{-1}\big)}. \tag{39}$$

A similar condition holds for buy sequences, only replacing $\prod_{i=1}^{t-1}(\delta_i^s)^{-1}$ with $\prod_{i=1}^{t-1} \delta_i^b$. Combining the initial spread and the spreads after consecutive sell orders in the benchmark and extended models (Eqs. (10), (28), and (39)) obtains $\Delta S_t = \Delta S_t^A + \Delta S_t^V$, where

$$\Delta S_t^V = \frac{-(\delta_1 - 1) \cdot \big(\delta_1^{-(t-1)} - 1\big)^2}{\big(\delta_1^{-(t-1)} + \delta_1\big) \cdot \big(\delta_1^{-(t-1)} + (\delta_1)^{-1}\big) \cdot \big(\delta_1 + 1\big)} < 0, \tag{40}$$

and

$$\Delta S_t^A = \frac{\prod_{i=1}^{t-1}(\delta_i^s)^{-1} \cdot \big(\delta_t^s - (\delta_t^b)^{-1}\big)}{\big(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \delta_t^s\big) \cdot \big(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + (\delta_t^b)^{-1}\big)} - \frac{\delta_1^{-(t-1)} \cdot (\delta_1 - \delta_1^{-1})}{(\delta_1^{-(t-1)} + \delta_1) \cdot (\delta^{-(t-1)} + \delta_1^{-1})} > 0. \tag{41}$$

Eq. (40) shows the downward pressure on the bid-ask spread due to the resolution of uncertainty about the fundamental value, whereas Eq. (41) shows the upward pressure due to learning about the adverse selection. Panel (b1) of Figure (3) illustrates that the upward pressure is due to the increase in the belief about the high proportion of informed trading π_t during consecutive sell orders. It follows from Eqs. (40) and (41) that the upward pressure due to learning about the adverse selection ΔS_t^A dominates the downward pressure due to learning about the fundamental value ΔS_t^V (i.e., $\Delta S_t^A + \Delta S_t^V > 0$) if and only if

$$\delta_1 < 1 + \frac{2 \cdot \big(\delta_t^s - (\delta_t^b)^{-1}\big)}{\Big[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s) \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\Big]}. \tag{42}$$

A similar condition holds for buy sequences. Eq. (42) shows that when the initial belief about the proportion of informed traders or the informativeness of orders δ_1 is sufficiently low, a continuous selling pressure leads to a wider spread (i.e., liquidity deterioration) relative to the initial spread. This is intuitive because order imbalance is not expected when the initial belief about the adverse selection (π_1 or δ_1) is sufficiently low. Thus, it presents a larger shock to a market maker, leading ΔS_t^A to dominate ΔS_t^V. Panel (b2) of Figure 3 illustrates the effects of multidimensional learning on the quotes of a market maker and contrasts it with the quotes of a market maker learning only about the fundamental value. The quotes are consistent with the empirical observations that the bid moves downward faster than the ask during the periods of large selling pressure (e.g., CFTC-SEC (2010a, 2010b)). Consequently, the faster reaction of the bid and the delayed reaction of the ask compared to that of the benchmark model lead the spread to widen in response to order imbalance as illustrated in Panel (b3).[11] The next proposition summarizes the role of learning about toxicity in stabilizing the market during balanced and destabilizing during unbalanced order flow.

Proposition 10. *In the presence of composition uncertainty;*

(i) the bid-ask spread is given by

$$S_{\alpha,t} = S_1 + \Delta S_t, \tag{43}$$

where S_1 is the initial bid-ask spread and ΔS_t is the (stabilizing for $\Delta S_t < 0$ and destabilizing for $\Delta S_t > 0$) liquidity distortion relative to the initial spread;

(ii) balanced order flows always stabilize the market;

(iii) sequences of sell orders destabilize the market if and only if

$$\delta_1 < 1 + \frac{2 \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \prod_{i=1}^{t-1} (\delta_i^s) \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\right]}; \tag{44}$$

(iv) sequences of buy orders destabilize the market if and only if

$$\delta_1 < 1 + \frac{2 \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1} \delta_i^b + \prod_{i=1}^{t-1} (\delta_i^b)^{-1} \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\right]}. \tag{45}$$

[11] In fact, flash crashes do not only occur on the way down. The sharp price rise in the price of a 10-year US Treasury security (37 bps. trading range) on 15 Oct. 2014, also known as a "Flash Rally", is a recent example of this in which the market functioned with a strained liquidity, a high volatility, and a high trading volume in the presence of excessive buy orders (e.g., U.S. Dept. of the Treasury et al. (2014)). Consecutive buy orders obtain symmetric results. In the case of 20 consecutive buys with the same parameter values, π_t increases in the same way it does in 20 consecutive sells since only the amount, not the direction, of order imbalance is informative about π_t. As the bid and ask prices increase with each buy order, the ask price moves upward faster compared to the delayed reaction of the bid price, leading to a wider bid-ask spread.

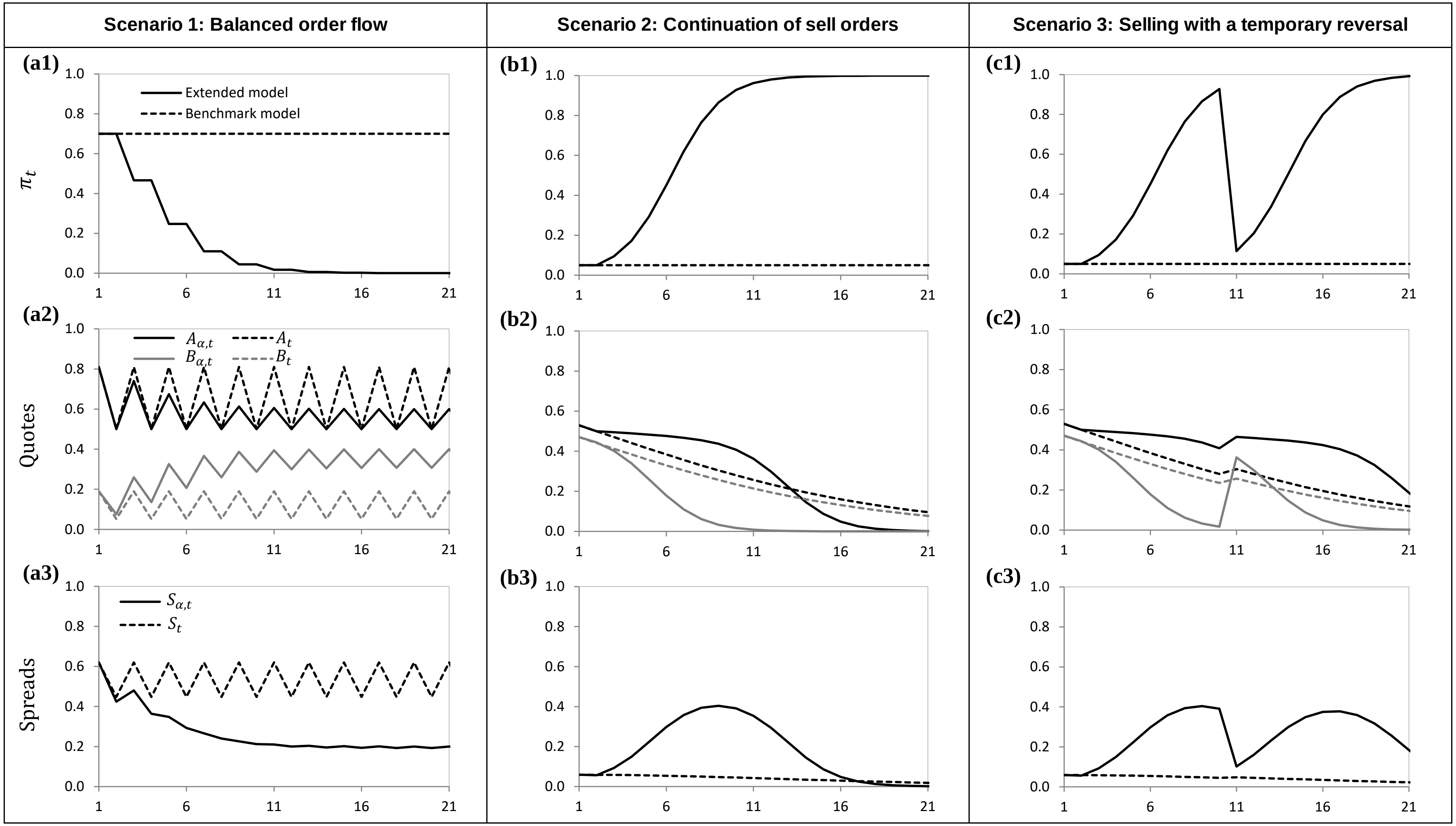

Figure 3. The dynamics of the market maker's belief about the adverse selection, quotes and bid-ask spreads.
Panels (a1)-(a3) plot the market maker's belief about the high informed trading, π_t, quotes, and bid-ask spread in the benchmark and extended models in the face of 20 perfectly balanced orders (i.e., a buy following a sell order). The parameter values for Panels (a1)-(a3) are $\alpha_L = 0.2$, $\alpha_H = 0.8$, and $\pi_1 = 0.7$. Panels (b1)-(b3) plot the same variables in the face of 20 consecutive sell orders (i.e., $N_t = -20$ at $t = 21$). Panels (c1)-(c3) plot the same variables in the face of 20 consecutive sell orders up to $t = 21$ with one reversal (buy) at t=10. The parameter values for Panels (b1)-(b3) and (c1)-c(3) are $\alpha_L = 0.01$, $\alpha_H = 0.99$, and $\pi_1 = 0.05$. Other parameter values are $p_1 = 0.5$ and $q = 1$.

Lastly, Panels (c1)-(c3) of Figure 3 illustrate the market maker's belief about the high proportion of informed traders, quotes, and spread during consecutive sell orders with a temporary buy reversal. In the benchmark model without learning about the adverse selection, a reversal in order flow makes the market less liquid as it increases uncertainty about the fundamental value (i.e., $\Delta S_t^V > 0$). In the presence of learning about the adverse selection, however, the standard prediction is also not necessarily true since a temporary reversal can substantially decrease the market maker's belief about the adverse selection, leading to downward pressure on the bid-ask spread (i.e., $\Delta S_t^A < 0$). When the gap between the low α_L and high α_H proportion of informed traders is sufficiently high, the downward pressure due to learning about the adverse selection ΔS_t^A dominates the upward pressure due to learning about the fundamental value ΔS_t^V, leading to a liquidity improvement. The dominating downward revision about the adverse selection (i.e., $\Delta S_t^A + \Delta S_t^V < 0$) and the resulting liquidity improvement is opposite to what is predicted by the standard models. Panel (c1) of Figure 3 illustrates the reduction in the market maker's belief about the high proportion of informed traders during a reversal and Panels (c2)-(c3) highlight a subsequent liquidity improvement.

5.2. **Sharp price movements.** In this subsection, we analyze the information content of trades in the presence of learning about adverse selection and its role in contributing to sharp price movements as another form of market instability. The main intuition that we want to rigorously characterize is that when the market maker revises her perceived informativeness of order flow, she gives more (resp. less) credit to past orders and prices adjust accordingly. This means when the market maker is uncertain about the proportion of informed traders, an order impacts the market maker's beliefs about the future value of the security in two ways. The first is simply that buys (resp. sells) increase the likelihood that the fundamental value is high (resp. low) (i.e., the standard price discovery effect). A second effect is that the market maker also updates her belief about the informativeness of order flow and then uses this new belief to reassess what she had learned from past order flow. We term the second effect "repricing history".[12]

To disentangle the standard price discovery and repricing history effects, we contrast the rational market maker's learning about the payoff with the learning of the myopic market maker

[12] The term repricing history shouldn't be confused by the fact that our notion of equilibrium requires that the market maker does not regret ex-post for the trades that she is obliged to make as in the Glosten-Milgrom type models. Moreover, the stochastic process (N_t, t) is Markov, meaning that the distribution of $(N_{t+1}, t+1)$ depends on only (N_t, t) and is independent of the history in our setting. This follows because the trades are independently and identically distributed and (N_t, t) are counting processes for trades and time.

that learns from order flow as if it is always the first order (myopically). We interpret the difference in the learning of the rational and myopic market makers about the fundamental value of the security as repricing history effect. In essence, our task is to determine the components of revision in belief about the security payoff (i.e., $\Delta p_t = p_{t+1} - p_t$) when the rational market maker learns about the payoff from p_t to p_{t+1} with an order D_t. We are interested in the contributions of the two components (standard price discovery and repricing history) of the rational market maker's learning about the payoff in the face of different order flow patterns. Various contributors to the market maker's learning about the payoff are useful in understanding large and sharp price movements in financial markets. Formally, the rational market maker's total revision in belief about the security payoff at time t follows from Eqs. (21) and (22) as

$$\Delta p_t = p_{t+1} - p_t = \begin{cases} \dfrac{p_t \cdot (1-p_t) \cdot \left(\delta_t^b - 1\right)}{1 + p_t \cdot \left(\delta_t^b - 1\right)}, & \text{if} \qquad D_t = +1, \\ \dfrac{p_t \cdot (1-p_t) \cdot \left((\delta_t^s)^{-1} - 1\right)}{1 + p_t \cdot \left((\delta_t^s)^{-1} - 1\right)}, & \text{if} \qquad D_t = -1. \end{cases} \tag{46}$$

To characterize the standard price discovery component, consider a myopic market maker who learns from each order as if it is always the first order (recall that initially nature independently chooses $\hat{\alpha}$ and $\hat{V}$). This means that she does not observe the history and only has posterior beliefs about the payoff $\hat{V} \in \{0,1\}$ and the proportion of informed traders $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$. Such a market maker learns from each order myopically (i.e., one step ahead) as if nature has just independently chosen $\hat{V}$ and $\hat{\alpha}$. By one step learning about the payoff as if it is the first order, the myopic market maker does not reassess her prior learning about the payoff (i.e., the information in past order flow). The next lemma characterizes the learning of the myopic market maker without repricing history, where we carry the original notation with a superscript m describing the myopic market maker.

Lemma 11. *Let the informativeness of orders be $\delta_t^m = \frac{1+(\pi_t^m \cdot \alpha_H + (1-\pi_t^m)\cdot \alpha_L)}{1-(\pi_t^m \cdot \alpha_H + (1-\pi_t^m)\cdot \alpha_L)}$, where π_t^m is given by*

$$\pi_t^m = \frac{\left(1 + \alpha_H \cdot (2 \cdot p_{t-1}^m - 1) \cdot D_{t-1}\right) \cdot \pi_{t-1}^m}{\left(1 + \left(\pi_{t-1}^m \cdot \alpha_H + (1-\pi_{t-1}^m) \cdot \alpha_L\right) \cdot (2 \cdot p_{t-1}^m - 1) \cdot D_{t-1}\right)}. \tag{47}$$

Let the geometric mean of informativeness of orders be $\bar{\delta}_t^m = \left(\prod_{i=1}^{t-1} \delta_i^m\right)^{\frac{1}{t-1}}$ and weighted order imbalance $\bar{N}_t = \sum_{\tau=1}^{t-1} D_\tau \cdot w_\tau$, where $w_\tau = \frac{(t-1)\cdot \ln \delta_\tau^m}{\sum_{i=1}^{t-1} \ln \delta_i^m}$. Then the dynamics of the expectations about the payoff satisfy

$$\frac{p_{t+1}^m}{1-p_{t+1}^m} = \frac{p_t^m}{1-p_t^m} \cdot (\delta_t^m)^{D_t}, \tag{48}$$

and hence

$$p_t^m = \frac{(\bar{\delta}_t^m)^{\bar{N}_t}}{1+(\bar{\delta}_t^m)^{\bar{N}_t}}. \tag{49}$$

Lemma 11 shows that the myopic market maker (as in the benchmark and extended models) revises her belief upward (resp. downward) with a buy (resp. sell) order and the revision is higher with high informativeness of orders δ_t^m. Assuming independence between $\hat{V}$ and $\hat{\alpha}$ reduces the learning of the rational market maker to that of the myopic market maker and further assuming $\alpha_L = \alpha_H$ reduces the myopic market maker's learning to the learning in the benchmark model. An interesting difference of the rational and the myopic market maker is that the myopic market maker evaluates the information content of buy and sell orders at a given point in time equally (footnote 9 shows how δ_t^b and δ_t^s reduce to symmetric δ_t^m), implying that the repricing history effect causes asymmetric informativeness of orders (δ_t^b and δ_t^s) characterized in Proposition 5. The myopic market maker's revision in belief about the payoff, $\Delta p_t^m = p_{t+1}^m - p_t^m$, at time t follows from Eq. (48) as

$$\Delta p_t^m = \frac{p_t^m \cdot (1-p_t^m) \cdot \big((\delta_t^m)^{D_t} - 1\big)}{1 + p_t^m \cdot \big((\delta_t^m)^{D_t} - 1\big)}, \tag{50}$$

where δ_t^m is determined by π_t^m. The repricing history effect emerges from the difference between rational belief revision about the fundamental value by considering the whole order flow history Δp_t and myopic learning with only the standard price discovery component Δp_t^m, i.e.,

$$\Delta p_t^r = \Delta p_t - \Delta p_t^m, \tag{51}$$

where Δp_t^r stands for the repricing history. The repricing history effect has important implications for the dynamics of informativeness and price impacts of orders, and therefore for beliefs about the fundamental value and prices, especially during highly unbalanced order flow.

First, it leads to asymmetric price reactions due to the differential information content of orders (i.e., $\delta_t^b \neq \delta_t^s$). The asymmetry ($\delta_t^b - \delta_t^s$) is more pronounced during highly unbalanced order flow. More precisely, as the amount of order imbalance increases the difference between informativeness of buy and sell orders increases following Eqs. (18) and (19) ($\delta_t^b > \delta_t^s$ for negative and $\delta_t^s > \delta_t^b$ for positive imbalance).[13] This is intuitive because a buy (resp. sell) order during high sell (resp. buy) imbalance leads the market maker to learn that the past order flow may not have been as informed, resulting in a quick reassessment of the prior learning about the payoff. This means the reversal in order flow is more informative than the continuation in order flow.[14] These results are absent in the myopic market maker since she treats each order as the first order (i.e., $\delta_t^b = \delta_t^s = \delta_t^m$) and the benchmark market maker with time-independent and symmetric informativeness of orders (i.e., $\delta_t^b = \delta_t^s = \delta$).

Second, the repricing history effect causes accelerating price impacts when the market maker receives continuation in order flow. Accelerating price impacts means the security price increases or decreases at an increasing rate. Intuitively, in the presence of uncertainty about the proportion of informed traders, a buy (resp. sell) after consecutive buys (resp. sells) not only signals that the asset value is high (resp. low), but also signals that the previous buys (sells) are more informed, leading to additional upward (resp. downward) revision in the market maker's belief about the fundamental value. The empirical literature mainly focuses on $I_t = \frac{|\Delta p_t|}{p_t}$ as a proxy for the price impact. The price impact of a sell order at time t follows from Eq. (46) as

$$I_t = \frac{|\Delta p_t|}{p_t} = \frac{(1 - p_t) \cdot \big(1 - (\delta_t^s)^{-1}\big)}{1 + p_t \cdot \big((\delta_t^s)^{-1} - 1\big)}, \tag{52}$$

which reduces to

$$I_t = \frac{1 - (\delta_t^s)^{-1}}{1 + \prod_{i=1}^{t} (\delta_i^s)^{-1}}, \tag{53}$$

during consecutive sell orders. In the benchmark model, the price impact of an order D_t at time t is given by $I_t = \frac{|\delta^{D_t} - 1|}{\delta^{N_{t+1}} + 1}$, which always attenuates with order imbalance (i.e., $I_t < I_{t-1}$). Unlike the benchmark model, Eq. (53) shows that the price impact can accelerate in our model with the continuation in order flow. More precisely, the price impact accelerates $I_t > I_{t-1}$ if and only if the informativeness of order is sufficiently large, i.e.,

$$\delta_t^s > 1 + \frac{\big(\delta_{t-1}^s - 1\big) \cdot \big(\prod_{i=1}^{t-1} \delta_i^s + 1\big)}{\delta_{t-1}^s + \prod_{i=1}^{t-1} \delta_i^s}. \tag{54}$$

[13] In fact, it is straightforward to show that $\lim_{t\to\infty} \delta_t^b = \frac{1+\alpha_L}{1-\alpha_H}$ and $\lim_{t\to\infty} \delta_t^s = \frac{1+\alpha_H}{1-\alpha_L}$, leading to the maximum asymmetry $\delta_t^b - \delta_t^s = \frac{\alpha_H^2 - \alpha_L^2}{(1-\alpha_H)\cdot(1-\alpha_L)}$ when the market maker observes infinite sequences of sell orders. The result is symmetric for an infinite sequence of buy orders.

[14] Of course, there might be other explanations for why continuations and reversals in order flow have different information content. For example, one obvious explanation might be the presence of history-dependent (e.g., positively correlated with the last trade) uninformed traders in the market so that the reversal is more likely associated with informed trading (e.g., Easley, Kiefer and O'Hara (1997)). While history-dependent uninformed trading can lead to the differential information content of continuations and reversals, it does not necessarily lead to the large price swings that we explain with our model.

This occurs because in the presence of uncertainty about the adverse selection, the price impact decreases with the belief about the payoff p_t as in the benchmark model (i.e., $\frac{\partial I_t}{\partial p_t} < 0$), but also increases with the informativeness of buy and sell orders (i.e., $\frac{\partial I_t}{\partial \delta_t^s} > 0$ and $\frac{\partial I_t}{\partial \delta_t^b} > 0$). Therefore, our model can explain accelerating price impacts similar to those observed during flash crashes (e.g., CFTC-SEC (2010a)), whereas the benchmark model is always associated with attenuating price impacts in response to order imbalance.

Sharp price movements in financial markets can arise as a consequence of these two effects (accelerating price impacts during continuations and more informative reversals) stemming from the repricing history effect. Compared to the myopic and benchmark market makers, the rational market maker in the presence of uncertainty about adverse selection learns faster in response to continuation in order flow (order imbalance) due to accelerating price impacts ($I_t > I_{t-1}$) and re-learns even faster in response to a reversal in order flow due to more informative reversals ($\delta_t^b > \delta_t^s$ during sell and $\delta_t^s > \delta_t^b$ during buy sequences). This generates a sharp decline or rise followed by a quick reversal in price, such as the typical price path during a flash crash or rally (e.g., CFTC (2010a, 2010b) and U.S. Dept. of the Treasury et al. (2014)).

We illustrate the implications of repricing history in Figure 4 by considering three order flow patterns: consecutive buys in Panels (a1)-(a3), temporary sells with a subsequent reversal in Panels (b1)-(b3), and consecutive buys with a temporary reversal in Panels (c1)-(c3). Panels (a1) and (a2) of Figure 4 show that during the buying pressure, the rational market maker in the presence of uncertainty about the adverse selection learns faster about the payoff and the repricing history effect contributes to faster learning compared to the myopic and benchmark market makers. Panel (a3) complements these findings by showing that the informativeness of a buy order at time t, δ_t^b, slightly increases compared to the initial informativeness δ_1, which stays at its initial level in the benchmark model. Intuitively, this is because with consecutive buy orders the repricing history effect causes accelerating price impacts as the market maker reassesses what can be learned from past buy orders, leading to faster learning. Panel (a3) additionally shows that the asymmetry between the information content of buy and sell orders increases with order imbalance. More precisely, it shows that the reversals (sells) become more informative as the market maker receives continuation in order flow (buys).

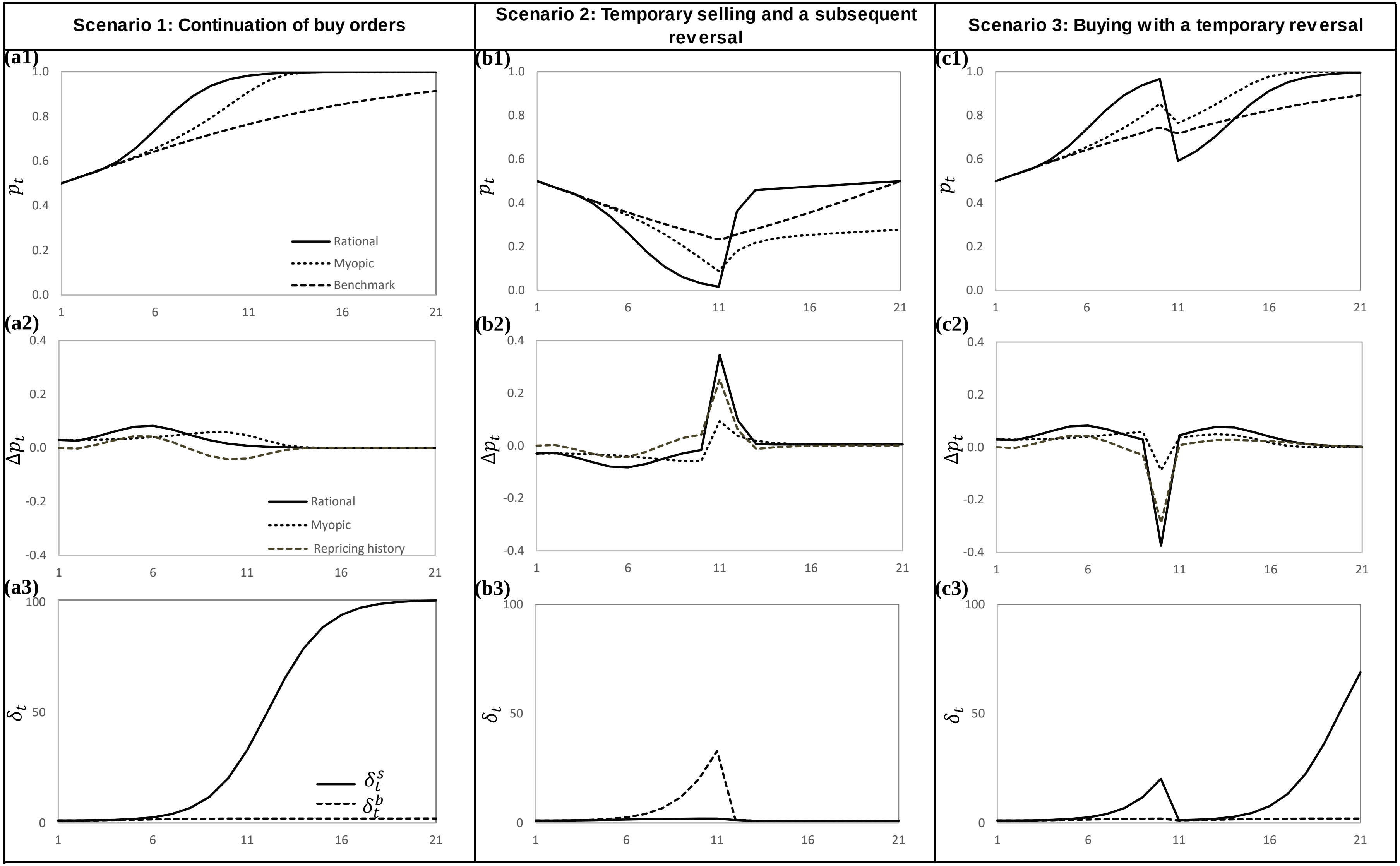

Figure 4. The implications of repricing history effect.
Panels (a1)-(a3) plot the three market makers' (rational, myopic and benchmark) beliefs about the payoff, p_t, contribution of the repricing history effect, Δp_t^r, and informativeness of buy and sell orders, δ_t^b and δ_t^s, in the face of consecutive buy orders up to $t = 21$. Panels (b1)-(b3) plot the same variables in the face of consecutive sells up to $t = 11$ followed by consecutive buys up to $t = 21$. Panels (c1)-(c3) plot the same variables in the face of consecutive buys up to $t = 21$ with one reversal (sell) at $t = 10$. The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q = 1$, $p_1 = 0.5$, and $\pi_1 = 0.05$.

Panel (b1) shows that during consecutive sells followed by consecutive buys, the conditional expected payoff p_t declines faster for the same reason it occurs during consecutive buys and reverses quickly due to more informative reversals (buys) illustrated in Panel (b3). Panel (b2) highlights that the magnitude of repricing history effect in contributing the reassessment of the market maker's beliefs about the fundamental value can be substantial during the reversal in order flow. The repricing history effect leads the informativeness of buy orders δ_t^b to be greater than informativeness of sell orders δ_t^s during the sell sequence. Thus, similar to flash crashes, the repricing history effect can generate sharp crashes due to accelerating price impacts and quick recoveries due to more informative reversals.

Finally, Panels (c1) and (c2) highlight the information content of one reversal (sell) during a buy sequence. With one sell order at $t = 10$, the market maker reassesses what she had learnt up to $t = 10$ by treating the previous buy orders as less informed than previously believed, because the sell causes a downward revision in belief about the high proportion of informed traders, whereas the benchmark market maker fails to take this into account and the myopic market maker only does so without reassessment of the full order flow history. Additionally, Panel (c3) illustrates that the informativeness of sell orders increases as the market maker faces continuation in buy orders, which is symmetric to continuation in sell orders.

6. Empirical Implications

In this section, we discuss the empirical implications of our model.

6.1. **Prevalence of flash crashes.** Anecdotal evidence suggests that "mini flash crashes" driven by large aggressive orders, e.g., intermarket sweep orders, occur nearly every day.[15] Our model offers two explanations for why this is the case. First, with the rise of algorithmic trading and availability of market data, the financial market ecosystem has now dramatically changed.[16] The composition of market participants has never been more complex and uncertain.[17] The technological developments have amplified the uncertainty in asymmetric information problem of the modern liquidity providers. In our model, this corresponds to the gap between the low

[15] Mini or micro flash crashes occur when a stock price spikes up or down in a small time frame. Nanex Research offers exhaustive documentation of "mini flash crashes": http://www.nanex.net/NxResearch/ResearchPage/3/.

[16] See, for instance, "The big changes in US markets since Black Monday" (Financial Times, October 19, 2017), "3 ways big data is changing financial trading" (Bloomberg, July 4, 2017).

[17] The new era of data revolution stimulated some of the data analytics firms to enter into a hedge fund business (e.g., Cargometrics). See, for instance, "When Silicon Valley came to Wall Street" (Financial Times, October 28, 2017), "Rise of quant: New hedge funds next year to embrace high tech" (Bloomberg, December 21, 2017).

α_L and the high α_H probability of informed trading. Indeed, as the gap between α_L and α_H increases our model shows that the market becomes more vulnerable to order imbalance and the magnitude of the instability caused by order imbalance increases. Small order imbalances with high composition uncertainty can lead to liquidity black holes, large price swings, elevated volatility, and consequently, the prevalence of flash crashes.

A second reason is associated with the increased competition among liquidity providers as a result of the proliferation of HFT and demise of the designated market makers. The current market structure incentivizes learning about the time-varying adverse selection risk to ensure spreads always reflect the level of toxicity. Therefore, efficient learning about the time-varying level of adverse selection is crucial for a liquidity provider to remain competitive in today's major equity markets. These effects can also contribute to the increased prevalence of flash crashes.

6.2. **Model predictions.** Our model makes a number of empirical predictions about the dynamics of prices, liquidity, and order flow. Some of these predictions provide a theoretical explanation for results that have been reported in the empirical market microstructure literature. Yet others are more nuanced empirical predictions that are yet to be tested, forming a foundation for future empirical analysis. The most straightforward prediction of our model is that during the selling (resp. buying) pressure the bid (resp. ask) moves faster than the ask (resp. bid), and therefore, liquidity evaporates (this effect is not possible in the benchmark model because sells during sell imbalance always impact the ask more than the bid and vice versa). This finding is consistent with the empirical results of Engle and Patton (2004) who find that sells impact the bid more than the ask, which stands in contrast to the theoretical results of Glosten and Milgrom (1985). Our model shows that this occurs because of the market maker's learning about toxicity (adverse selection) from order flow. A cursory examination of the transactions data series of E-mini and SPY (S&P 500 ETF) confirms that the same phenomenon was present during the May 2010 Flash Crash. Second, the increasing informativeness of orders and wider spreads in response to order imbalances imply that the trades that arrive when the spread is wide have a greater price impact. This is consistent with Hasbrouck (1991), who finds that trades that occur in the face of wider spreads have a larger price impact than those that occur when the spreads are narrow.

Our model also develops several other empirical implications about the dynamics of spreads, informativeness, and price impacts of trades during various order flow patterns. The model predicts that liquidity can improve during balanced orders and reversals in order flow due to learning about the adverse selection. While these results are intuitive in the presence of learning about the adverse selection, both are opposite to what is predicted by standard market microstructure models. In addition, the informativeness and price impacts of trades are time varying and asymmetric due to learning about the adverse selection. The model predicts that the asymmetry in price impacts of buy and sell orders increases (resp. decreases) as order imbalance increases (resp. decreases).

In our model, accelerating price impacts and more informative reversals during unbalanced order flow naturally arise due to the repricing history effect. Engle and Patton (2004) find strong evidence on the differential impacts of buy and sell orders on the bid and ask prices. They interpret the result with potentially multiple information events per day. Our model shows that uncertainty about the proportion of informed traders, the quality of their signals, multidimensional learning, and consequently, the repricing history effect are what drive this asymmetry. The accelerating price impact and more informative reversals due to the repricing history effect can generate a security price dynamics similar to flash crashes.

Overall, our results suggest that financial markets are more susceptible to instability in response to order imbalance in times when adverse selection is believed to be low and can digest more imbalance in times when adverse selection is high. This follows because order imbalance destabilizes the market when the initial belief about the adverse selection is sufficiently low, which we also show occurs after balanced order flow.

7. Model Discussion and Extensions

In this paper, we use a simple sequential trading model in the sense of Glosten and Milgrom (1985) to provide intuition about the destabilizing role of order imbalances in financial markets and the occurrences of financial crashes in the absence of the fundamental news about the security value. An interesting question is how sensitive our results are to our modeling approach. In this section, we address this by considering how some extensions and generalizations of our model affect the results.

7.1. **Other sources of uncertainty about adverse selection.** The model presented thus far incorporates uncertainty about the proportion of informed traders and the security payoff. Allowing uncertainty in the other dimension of adverse selection problem complicates the notation, but does not affect our results. For example, in Appendix A, we allow the market maker to be uncertain about the different combinations of uncertainties in the proportion of informed traders, the quality of their private information, and the security payoff. We show that while the magnitude of market instability may change, the main qualitative results of our model are robust.

7.2. **Endogenous uninformed trading.** For convenience, we assumed that the motivation for uninformed trading is exogenous. There may be other uninformed traders whose demands reflect more complex motivations resulting in distributions of private valuations that drive their trading decisions (e.g., Easley et al. (1997), Glosten and Putnins (2016)). In fact, if uninformed traders have elastic demands sensitive to trading costs, the destabilizing effects of order imbalances are also amplified. To see this, suppose there are uninformed traders who have a distribution of private values and there is occasionally a highly impatient trader that is either an informed trader or a distressed uninformed trader (one that has a private valuation very far from the current price). In normal conditions, when the spread is tight, the market maker receives some order flow from the uninformed traders (balanced) and some of the imbalance from the informed or distressed uninformed traders. If the imbalance is sufficiently strong (that is, the desperate or highly informed trader hits the market too aggressively), the change in quotes is sufficiently large (due to updating beliefs about the probability of informed trading). This scares off most of the uninformed traders and makes the order flow even more unbalanced, causing a feedback loop that can amplify the destabilizing effects of order imbalance. Thus our modeling of uninformed traders as exogenous and insensitive to the cost of trading is conservative in that it understates the severity of the impact of order imbalance.

8. Conclusion

With increasing competition between liquidity providers (e.g., due to endogenous liquidity providers) and the use of algorithms in trading, market participants learn not only about the fundamental values of assets, but also other characteristics of markets that are important for extracting information from order flow, such as the degree of adverse selection. Such multidimensional learning can have very different implications for trading behavior, market liquidity, and stability compared to learning only about the fundamental values of assets. In this paper, we explore the effects of order imbalance when liquidity providers learn not only about the fundamental value of the asset, as in the standard market microstructure models, but also about the proportion of informed traders and the quality of their information. The multidimensional learning explains a variety of empirical regularities not captured by the standard asymmetric information models of market microstructure theory.

Our theoretical model with additional learning about the toxicity of order flow shows the potentially destabilizing effects of order imbalance. We show that order imbalance can have a stabilizing effect on the market by narrowing the spread because it reduces uncertainty about the fundamental value and destabilizes the market by widening the spread because it increases belief about the high adverse selection risk. The destabilizing effect of order imbalance dominates its stabilizing effect when the initial belief about the adverse selection is sufficiently low. This means financial markets become more susceptible to imbalance-induced instability when the past order flow is more balanced. Put differently, in our setting, it is the order imbalance during stability that leads to instability.

In addition to the sudden liquidity dry-ups, order imbalance can also naturally lead to a sharp price decline and a quick recovery similar to flash crashes due to the "repricing history" effect. We show that a sharp price decline occurs due to accelerating price impacts with continuations in order flow and a quick recovery occurs due to more informative reversals in order flow, both stemming from the "repricing history" effect. Overall, our model provides a theoretical framework for further empirical work characterizing the dynamics of order flow, liquidity, and prices.

APPENDIX A: EXTENSION TO MULTIPLE DIMENSIONS OF UNCERTAINTY

In this Appendix, we extend our model to allow the market maker to be uncertain about the security payoff, proportion of informed traders, and quality of informed traders' information.

8.1. **Uncertainty about the proportion of informed traders and quality of their signals.** Similar to Section 4, we assume that the probability of informed trading takes either low or high values from the set $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ with an initial prior probability of $\Pr(\hat{\alpha} = \alpha_H) = \pi_1$, where $0 < \alpha_L < \alpha_H < 1$ and $0 < \pi_1 < 1$. In addition, to extend our results to additional uncertainty about the quality of informed traders' information, we assume that the quality of their signals takes either low or high values from the set $\hat{q} \in \{q_L, q_H\}$ with an initial probability of $\Pr(\hat{q} = q_H) = \rho_1$, where $0.5 < q_L < q_H \leq 1$ and $0 < \rho_1 < 1$. We denote the market maker's belief about the high probability of informed trading conditional on the trading history as $\pi_t = \Pr\{\hat{\alpha} = \alpha_H | h_t\}$ and the informed traders having high-quality private information conditional on the trading history as $\rho_t = \Pr\{\hat{q} = q_H | h_t\}$. With two possible values for the future security payoff, probability of informed trading, and quality of their private information (i.e., $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$, $\hat{q} \in \{q_L, q_H\}$, $\hat{V} \in \{0, 1\}$), there are 8 possible disjoint states in this model. Denote the states $\mathcal{S} \in \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, where

$$\begin{aligned}
s_1 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_H, \hat{V} = 1\}, &\quad s_2 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_L, \hat{V} = 1\},\\
s_3 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_H, \hat{V} = 1\}, &\quad s_4 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_L, \hat{V} = 1\},\\
s_5 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_H, \hat{V} = 0\}, &\quad s_6 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_L, \hat{V} = 0\},\\
s_7 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_H, \hat{V} = 0\}, &\quad s_8 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_L, \hat{V} = 0\}.
\end{aligned} \tag{A-1}$$

The market maker's beliefs about the future security payoff, proportion of informed traders, and quality of their signals follow from Eq. (A-1), respectively, as

$$p_t = \Pr(\hat{V} = 1 | h_t) = \Pr(s_1 | h_t) + \Pr(s_2 | h_t) + \Pr(s_3 | h_t) + \Pr(s_4 | h_t), \tag{A-2}$$

$$\pi_t = \Pr(\hat{\alpha} = \alpha_H | h_t) = \Pr(s_1 | h_t) + \Pr(s_2 | h_t) + \Pr(s_5 | h_t) + \Pr(s_6 | h_t), \tag{A-3}$$

$$\rho_t = \Pr(\hat{q} = q_H | h_t) = \Pr(s_1 | h_t) + \Pr(s_3 | h_t) + \Pr(s_5 | h_t) + \Pr(s_7 | h_t). \tag{A-4}$$

The concept of equilibrium is the same as Definition 1. The next proposition characterizes the equilibrium quotes and bid-ask spread in the presence of multiple dimensions of uncertainty.

Proposition A1. *The equilibrium bid and ask prices in the presence of multidimensional uncertainty (i.e., future security payoff, proportion of informed traders, and quality of their signals) are respectively given by*

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1-p_t)}, \qquad \textit{and} \qquad A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1-p_t)}, \tag{A-5}$$

and the bid-ask spread is given by

$$S_{\alpha,t} = \frac{p_t \cdot (1-p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\big(p_t + \delta_t^s \cdot (1-p_t)\big) \cdot \big(p_t + (\delta_t^b)^{-1} \cdot (1-p_t)\big)}, \tag{A-6}$$

where

$$\delta_t^b = \frac{(1+\alpha_H(2q_H-1)) \cdot \Pr(s_1|h_t) + (1+\alpha_H(2q_L-1)) \cdot \Pr(s_2|h_t) + (1+\alpha_L(2q_H-1)) \cdot \Pr(s_3|h_t) + (1+\alpha_L(2q_L-1)) \cdot \Pr(s_4|h_t)}{(1-\alpha_H(2q_H-1)) \cdot \Pr(s_5|h_t) + (1-\alpha_H(2q_L-1)) \cdot \Pr(s_6|h_t) + (1-\alpha_L(2q_H-1)) \cdot \Pr(s_7|h_t) + (1-\alpha_L(2q_L-1)) \cdot \Pr(s_8|h_t)} \cdot \left(\frac{1-p_t}{p_t}\right), \tag{A-7}$$

and

$$\delta_t^s = \frac{(1+\alpha_H(2q_H-1)) \cdot \Pr(s_5|h_t) + (1+\alpha_H(2q_L-1)) \cdot \Pr(s_6|h_t) + (1+\alpha_L(2q_H-1)) \cdot \Pr(s_7|h_t) + (1+\alpha_L(2q_L-1)) \cdot \Pr(s_8|h_t)}{(1-\alpha_H(2q_H-1)) \cdot \Pr(s_1|h_t) + (1-\alpha_H(2q_L-1)) \cdot \Pr(s_2|h_t) + (1-\alpha_L(2q_H-1)) \cdot \Pr(s_3|h_t) + (1-\alpha_L(2q_L-1)) \cdot \Pr(s_4|h_t)} \cdot \left(\frac{p_t}{1-p_t}\right), \tag{A-8}$$

show the informativeness of buy and sell orders respectively. In addition, δ_t^b and δ_t^s are always greater than unity and increase with the proportions of informed trading α_L and α_H, and the qualities of the informed traders' signals q_L and q_H.

Proof. The proof follows similar to the proof of Proposition 5 (see Appendix B). The difference is to recognize that

$$B_{\alpha,t} = \Pr(\hat{V}=1|h_t, D_t=-1) = \Pr(s_1|h_t, D_t=-1) + \Pr(s_2|h_t, D_t=-1) + \Pr(s_3|h_t, D_t=-1) + \Pr(s_4|h_t, D_t=-1), \tag{A-9}$$

$$A_{\alpha,t} = \Pr(\hat{V}=1|h_t, D_t=-1) = \Pr(s_1|h_t, D_t=+1) + \Pr(s_2|h_t, D_t=+1) + \Pr(s_3|h_t, D_t=+1) + \Pr(s_4|h_t, D_t=+1), \tag{A-10}$$

which follow from the straightforward application of Bayes' rule, i.e., $\Pr(s_1|h_t, D_t=-1) = \frac{Pr(D_t=-1|s_1)}{\sum_{s_i \in S} \Pr(D_t=-1|s_i) \cdot \Pr(s_i|h_t)} \cdot \Pr(s_1|h_t)$, with the similar rules for the other states. The results follow once the probabilities of buy ($D_t=+1$) and sell ($D_t=-1$) orders in each state are calculated as

$$\begin{aligned} &\Pr(D_t|s_1) = \frac{1+\alpha_H(2q_H-1)\cdot D_t}{2}, \quad \Pr(D_t|s_2) = \frac{1+\alpha_H(2q_L-1)\cdot D_t}{2}, \quad \Pr(D_t|s_3) = \frac{1+\alpha_L(2q_H-1)\cdot D_t}{2}, \quad \Pr(D_t|s_4) = \frac{1+\alpha_L(2q_L-1)\cdot D_t}{2}, \\ &\Pr(D_t|s_5) = \frac{1-\alpha_H(2q_H-1)\cdot D_t}{2}, \quad \Pr(D_t|s_6) = \frac{1-\alpha_H(2q_L-1)\cdot D_t}{2}, \quad \Pr(D_t|s_7) = \frac{1-\alpha_L(2q_H-1)\cdot D_t}{2}, \quad \Pr(D_t|s_8) = \frac{1-\alpha_L(2q_L-1)\cdot D_t}{2}. \end{aligned} \tag{A-11}$$

■

Proposition A1 maintains the forms of the bid and ask in the benchmark and extended models with composition uncertainty (see Propositions 2 and 5). The only difference in this Appendix is that uncertainty about the quality of informed traders' information and the market maker's learning about it also affect the informativeness of orders, δ_t^b and δ_t^s, and consequently, the quotes and spread. One way to see the direct impact of uncertainty about the quality of traders' signals is to consider a myopic market maker, which leads to the symmetric information content of a buy and a sell

$$\delta_t^b = \delta_t^s = \frac{1 + \Big(\pi_t \cdot \alpha_H + (1-\pi_t) \cdot \alpha_L\Big) \cdot \Big(2 \cdot \big(\rho_t \cdot q_H + (1-\rho_t) \cdot q_L\big) - 1\Big)}{1 - \Big(\pi_t \cdot \alpha_H + (1-\pi_t) \cdot \alpha_L\Big) \cdot \Big(2 \cdot \big(\rho_t \cdot q_H + (1-\rho_t) \cdot q_L\big) - 1\Big)}. \tag{A-12}$$

Eq. (A-12) shows that the increase in ρ_t implies a higher adverse selection risk for the market maker, leading to a wider bid-ask spread. As Proposition A1 maintains the forms of bid and ask quotes, by the similar arguments, the dynamics of belief about the payoff, p_t, after each order D_t and for the sequences of buy or sell orders follow the same way outlined in Lemma 6 and Corollary 9, respectively. In addition, the market maker's learning about uncertainties follows analogous to Corollary 8.

Proposition A2. *In the presence of multidimensional uncertainty (i.e., future security payoff, proportion of informed traders, and quality of their signals);*

(i) *the market maker observing balanced order flows (i.e., $N_t = 0$ at time t) learns nothing about the security payoff (i.e., $p_t = p_1$), but revises her beliefs about the high informed trading and the informed traders having high-quality information downward (i.e., $\pi_t < \pi_1$, $\rho_t < \rho_1$).*

(ii) *the market maker observing sequences of buy or sell orders (i.e., $N_t = t-1$ or $N_t = -(t-1)$ at time t) revises her beliefs about the high informed trading and the informed traders having high-quality information upward (i.e., $\pi_t > \pi_1$, $\rho_t > \rho_1$).*

We now turn our attention to how beliefs about the future security payoff, proportion of informed traders, and quality of their signals evolve when there are sequences of buy and sell orders and their effects on the quotes and spread. Figures 5 and 6 illustrate the results in the presence of consecutive sell and buy orders, respectively. Panels (a)-(c) of the figures show that beliefs about the high informed trading and high-quality signals are revised upward stronger in the presence of multiple sources of uncertainty. The stronger upward revisions of beliefs about the high informed trading and high-quality signals result in high informativeness of orders, leading to a wider bid-ask spread first, but at the same time faster convergence due to faster learning about the payoff. Panels (d)-(f) are consistent with the practice that the destabilizing role of order imbalances is stronger in the presence of multiple dimensions of uncertainty about the adverse selection.

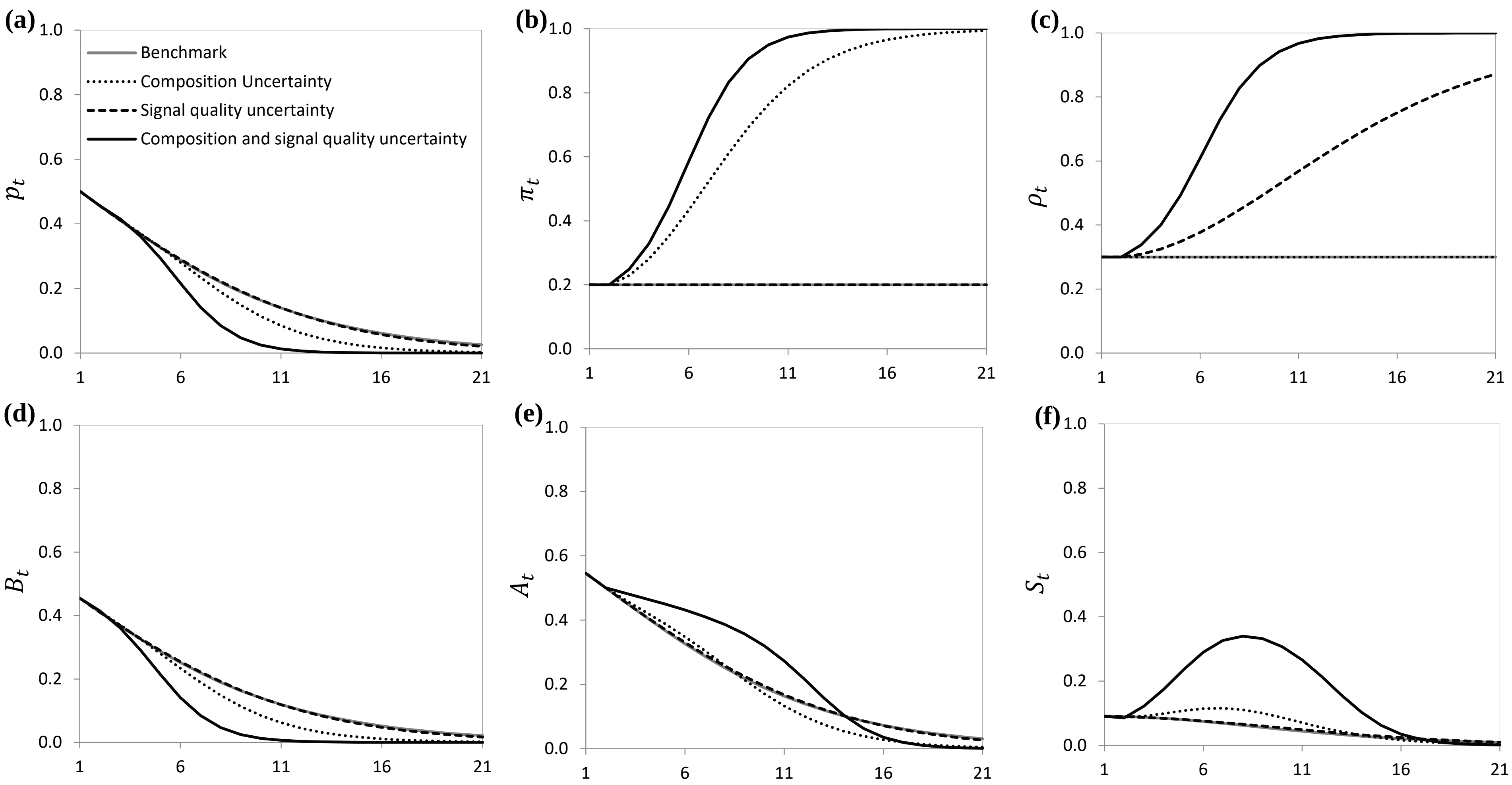

Figure 5. The dynamics of beliefs, quotes and bid-ask spread during the sequences of sell orders.
Panel (a) plots the market maker's belief about the payoff, p_t, (b) plots belief about the high informed trading, π_t, (c) plots belief about the informed traders having high quality information, ρ_t, (d) plots bid, B_t, (e) plots ask, A_t, and (f) plots bid-ask spread S_t in the face of 20 consecutive sell orders (i.e., $N_t = -20$ at $t = 21$). The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q_H = 1$, $q_L = 0.6$, $p_1 = 0.5$, $\pi_1 = 0.1$, and $\rho_1 = 0.3$.

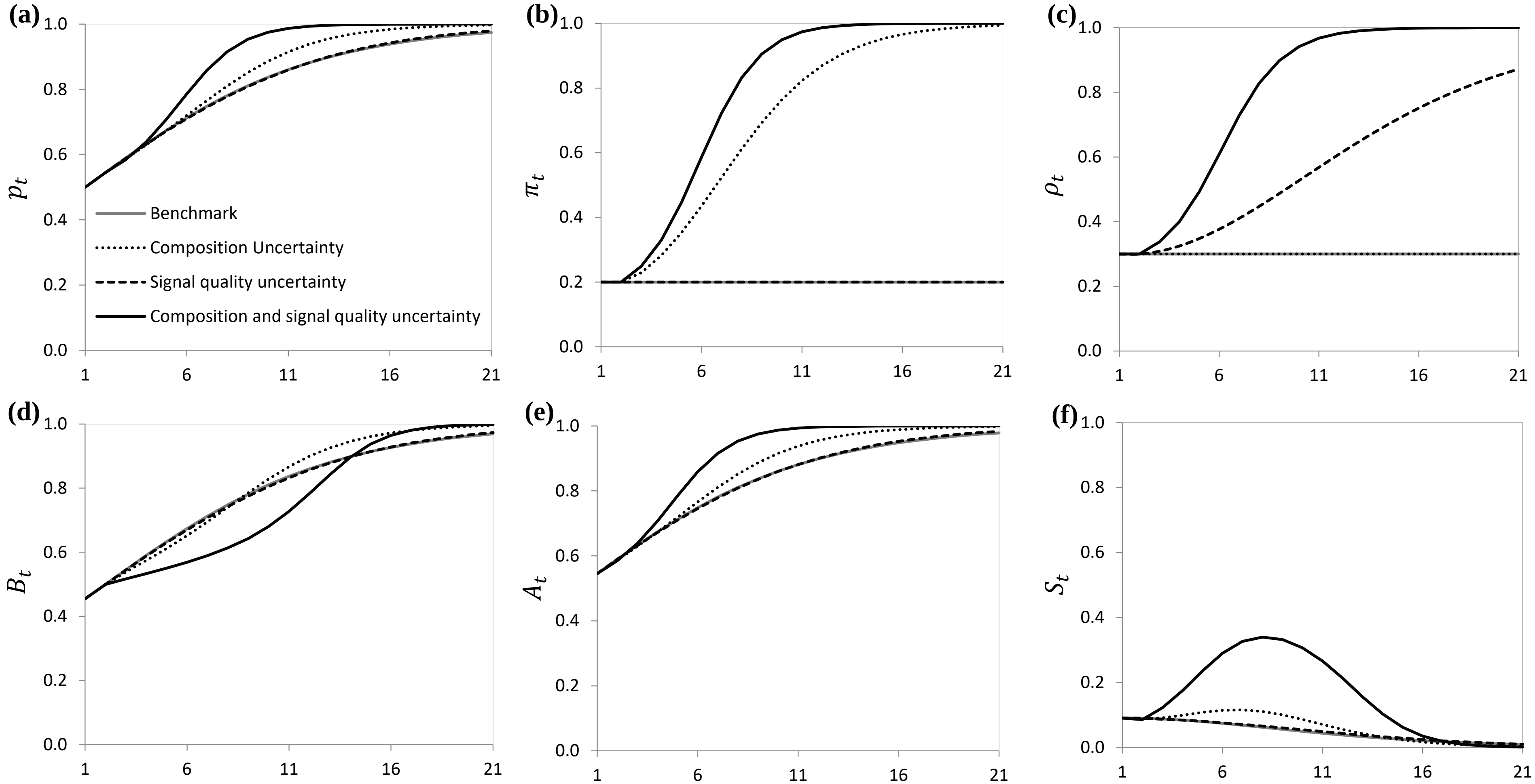

Figure 6. The dynamics of beliefs, quotes and bid-ask spread during the sequences of buy orders.
Panel (a) plots the market maker's belief about the payoff, p_t, (b) plots belief about the high informed trading, π_t, (c) plots belief about the informed traders having high quality information, ρ_t, (d) plots bid, B_t, (e) plots ask, A_t, and (f) plots bid-ask spread S_t in the face of 20 consecutive buy orders (i.e., $N_t = 20$ at $t = 21$). The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q_H = 1$, $q_L = 0.6$, $p_1 = 0.5$, $\pi_1 = 0.1$, and $\rho_1 = 0.3$.

APPENDIX B: PROOFS

Proof of Proposition 2. The following expressions follow from Bayes' theorem.

$$\Pr\{D_t = +1|\hat{V} = 1, h_t\} = \frac{1 + \alpha \cdot (2 \cdot q - 1)}{2}; \tag{B-1}$$

$$\Pr\{D_t = -1|\hat{V} = 1, h_t\} = \frac{1 - \alpha \cdot (2 \cdot q - 1)}{2}; \tag{B-2}$$

$$\Pr\{D_t = +1|h_t\} = \frac{1 + \alpha \cdot (2 \cdot p_t - 1) \cdot (2 \cdot q - 1)}{2}; \tag{B-3}$$

$$\Pr\{D_t = -1|h_t\} = \frac{1 + \alpha \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2}. \tag{B-4}$$

From conditions (1) and (3) of Definition 1 (i.e., the zero-expected-profit and Bayesian conditions) the bid and ask prices follow, respectively, as

$$B_t = E[\hat{V} = 1|h_t, D_t = -1] = \underbrace{\Pr\{V = 1|h_t\}}_{p_t} \cdot \frac{\Pr\{D_t = -1|\hat{V} = 1, h_t\}}{\Pr\{D_t = -1|h_t\}}, \tag{B-5}$$

$$A_t = E[\hat{V} = 1|h_t, D_t = +1] = \Pr\{V = 1|h_t\} \cdot \frac{\Pr\{D_t = +1|\hat{V} = 1, h_t\}}{\Pr\{D_t = +1|h_t\}}. \tag{B-6}$$

Substituting Eqs. (B-2) and (B-4) into Eq. (B-5) and Eqs. (B-1) and (B-3) into Eq. (B-6), defining

$$\delta = \frac{1 + \alpha \cdot (2 \cdot q - 1)}{1 - \alpha \cdot (2 \cdot q - 1)} > 1, \tag{B-7}$$

for $q \in (1/2, 1]$ and $\alpha \in (0, 1)$ and rearranging yields

$$B_t = \frac{p_t}{p_t + \delta \cdot (1 - p_t)}, \tag{B-8}$$

$$A_t = \frac{p_t}{p_t + \delta^{-1} \cdot (1 - p_t)}. \tag{B-9}$$

The bid-ask spread S_t follows from the difference of the ask in Eq. (B-9) and the bid in Eq. (B-8) as

$$S_t = \frac{p_t \cdot (1 - p_t) \cdot (\delta - \delta^{-1})}{\big(p_t + \delta \cdot (1 - p_t)\big) \cdot \big(p_t + \delta^{-1} \cdot (1 - p_t)\big)}. \tag{B-10}$$

Finally, differentiating Eq. (B-7) with respect to (w.r.t.) α and q obtains

$$\frac{\partial \delta}{\partial \alpha} = \frac{2 \cdot (2 \cdot q - 1)}{\big(1 - \alpha \cdot (2 \cdot q - 1)\big)^2} > 0 \quad \text{and} \quad \frac{\partial \delta}{\partial q} = \frac{4 \cdot \alpha}{\big(1 - \alpha \cdot (2 \cdot q - 1)\big)^2} > 0 \tag{B-11}$$

for $q \in (1/2, 1]$ and $\alpha \in (0, 1)$.

■

Proof of Lemma 3. Rearranging Eqs. (B-8) and (B-9) obtains

$$\frac{B_t}{1-B_t} = \delta^{-1} \cdot \frac{p_t}{1-p_t}, \tag{B-12}$$

$$\frac{A_t}{1-A_t} = \delta \cdot \frac{p_t}{1-p_t}. \tag{B-13}$$

Since $p_{t+1} = E_{t+1}[\hat{V}|h_t, D_t]$ is B_t if $D_t = -1$ and A_t if $D_t = 1$, it follows that

$$\frac{p_{t+1}}{1-p_{t+1}} = \begin{cases} \dfrac{B_t}{1-B_t}, & \text{if} \quad D_t = -1, \\ \dfrac{A_t}{1-A_t}, & \text{if} \quad D_t = +1. \end{cases} \tag{B-14}$$

Therefore,

$$\frac{p_{t+1}}{1-p_{t+1}} = \delta^{D_t} \cdot \frac{p_t}{1-p_t}. \tag{B-15}$$

Iterating from the first trade at $t = 1$ yields

$$\frac{p_t}{1-p_t} = \left(\frac{p_1}{1-p_1}\right) \cdot \delta^{(D_1+\ldots+D_{t-1})} = \left(\frac{p_1}{1-p_1}\right) \cdot \delta^{N_t}, \tag{B-16}$$

where p_1 is the initial prior probability and N_t is the order imbalance up to (but not including) the trade at time t. Solving Eq. (B-16) for p_t obtains

$$p_t = \frac{p_1 \cdot \delta^{N_t}}{1 + p_1 \cdot (\delta^{N_t} - 1)}, \tag{B-17}$$

and inserting the initial prior probability $p_1 = 0.5$ into Eq. (B-17) obtains

$$p_t = \frac{\delta^{N_t}}{1 + \delta^{N_t}}. \tag{B-18}$$

■

Proof of Corollary 4. *(i)* $p_t = p_1$ follows immediately from substituting $N_t = 0$ into Eq. (B-17). *(ii)* follows from the partial derivative of Eq. (B-17) w.r.t. N_t,

$$\frac{\partial p_t}{\partial N_t} = \frac{\delta^{N_t} \cdot (\ln \delta) \cdot p_1 \cdot (1-p_1)}{\left(1 + p_1 \cdot (\delta^{N_t} - 1)\right)^2} > 0, \tag{B-19}$$

which shows that p_t increases with positive order imbalance and decreases with negative order imbalance. In addition, the magnitude of increase or decrease is higher with more informative trades following

$$\frac{\partial p_t}{\partial \delta} = \frac{N_t \cdot \delta^{N_t - 1} \cdot p_1 \cdot (1 - p_1)}{\big(1 + p_1 \cdot (\delta^{N_t} - 1)\big)^2}, \tag{B-20}$$

which is greater than 0 for $N_t > 0$ and less than 0 for $N_t < 0$.

(iii) Substituting Eq. (B-17) into Eq. (B-10), we obtain

$$S_t = \frac{\delta^{N_t} \cdot (\delta - \delta^{-1}) \cdot p_1 \cdot (1 - p_1)}{\big(p_1 \cdot \delta^{N_t} + \delta \cdot (1 - p_1)\big) \cdot \big(p_1 \cdot \delta^{N_t} + \delta^{-1} \cdot (1 - p_1)\big)}. \tag{B-21}$$

It follows from Eq. (B-21) that at $t = 1$, the spread is given by

$$S_1 = \frac{(\delta - \delta^{-1}) \cdot p_1 \cdot (1 - p_1)}{\big(p_1 + \delta \cdot (1 - p_1)\big) \cdot \big(p_1 + \delta^{-1} \cdot (1 - p_1)\big)}. \tag{B-22}$$

Multiplying and dividing the right hand side of Eq. (B-21) with Eq. (B-22) yields

$$S_t = S_1 \cdot \frac{\big(p_1 + \delta \cdot (1 - p_1)\big) \cdot \big(p_1 + \delta^{-1} \cdot (1 - p_1)\big) \cdot \delta^{N_t}}{\big(p_1 \cdot \delta^{N_t} + \delta \cdot (1 - p_1)\big) \cdot \big(p_1 \cdot \delta^{N_t} + \delta^{-1} \cdot (1 - p_1)\big)}. \tag{B-23}$$

$S_t = S_1$ then follows from Eq. (B-23) when $N_t = 0$. When $p_1 = 0.5$, $S_1 = \frac{\delta - 1}{\delta + 1}$, which turns out to be the maximum spread since

$$\frac{\partial S_t}{\partial N_t} = \frac{(\delta - \delta^{-1}) \cdot \delta^{N_t} \cdot (\ln \delta) \cdot (1 - \delta^{2 \cdot N_t})}{\Big((\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})\Big)^2} = 0, \tag{B-24}$$

and $\frac{\partial^2 S_t}{\partial N_t^2} < 0$ when $N_t = 0$.

(iv) follows from Eq. (B-24) that $\frac{\partial S_t}{\partial N_t} < 0$ if $N_t > 0$ and $\frac{\partial S_t}{\partial N_t} > 0$ if $N_t < 0$, which shows that the spread narrows with order imbalance. Finally, the spread converges to zero as order imbalance goes to infinity following $\lim_{N_t \to +\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})} = \lim_{N_t \to +\infty} \frac{(\delta - \delta^{-1})}{\left(\delta^{N_t} + \delta^{-N_t} + \delta + \delta^{-1}\right)} = 0$ and $\lim_{N_t \to -\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})} = \lim_{N_t \to -\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{\left(\delta^{2 \cdot N_t} + \delta^{N_t - 1} + \delta^{N_t + 1} + 1\right)} = 0$ since $\delta > 1$.

■

Proof of Proposition 5. In the presence of composition uncertainty, the probability of an order D_t in each state is given by

$$\begin{aligned} \Pr(D_t|s_1) &= \frac{1 + \alpha_H \cdot D_t}{2}, \qquad & \Pr(D_t|s_2) &= \frac{1 - \alpha_H \cdot D_t}{2}, \\ \Pr(D_t|s_3) &= \frac{1 + \alpha_L \cdot D_t}{2}, \qquad & \Pr(D_t|s_4) &= \frac{1 - \alpha_L \cdot D_t}{2}, \end{aligned} \tag{B-25}$$

and the probability of an order D_t conditional on the trading history h_t is given by

$$\Pr(D_t|h_t) = \sum_{s_i \in \mathcal{S}} \Pr(D_t|s_i) \cdot \Pr(s_i|h_t), \qquad i = 1, 2, 3, 4. \tag{B-26}$$

The bid and ask quotes respectively follow as

$$\begin{aligned} B_{\alpha,t} &= E[\hat{V} = 1|h_t, D_t = -1] = \Pr(\hat{V} = 1|h_t, D_t = -1) \\ &= \Pr\{s_1|h_t, D_t = -1\} + \Pr\{s_3|h_t, D_t = -1\} \\ &= \frac{\Pr(D_t = -1|s_1, h_t)}{\Pr(D_t = -1|h_t)} \cdot \Pr(s_1|h_t) + \frac{\Pr(D_t = -1|s_3, h_t)}{\Pr(D_t = -1|h_t)} \cdot \Pr(s_3|h_t), \end{aligned} \tag{B-27}$$

$$\begin{aligned} A_{\alpha,t} &= E[\hat{V} = 1|h_t, D_t = 1] = \Pr(\hat{V} = 1|h_t, D_t = 1) \\ &= \Pr\{s_1|h_t, D_t = 1\} + \Pr\{s_3|h_t, D_t = 1\} \\ &= \frac{\Pr(D_t = 1|s_1, h_t)}{\Pr(D_t = 1|h_t)} \cdot \Pr(s_1|h_t) + \frac{\Pr(D_t = 1|s_3, h_t)}{\Pr(D_t = 1|h_t)} \cdot \Pr(s_3|h_t). \end{aligned} \tag{B-28}$$

Substituting Eqs. (B-25) and (B-26) into Eqs. (B-27) and (B-28), defining

$$\delta_t^b = \left(\frac{(1 + \alpha_H) \cdot \Pr(s_1|h_t) + (1 + \alpha_L) \cdot \Pr(s_3|h_t)}{(1 - \alpha_H) \cdot \Pr(s_2|h_t) + (1 - \alpha_l) \cdot \Pr(s_4|h_t)} \right) \cdot \left(\frac{1 - p_t}{p_t} \right) > 1, \tag{B-29}$$

$$\delta_t^s = \left(\frac{(1 + \alpha_H) \cdot \Pr(s_2|h_t) + (1 + \alpha_L) \cdot \Pr(s_4|h_t)}{(1 - \alpha_H) \cdot \Pr(s_1|h_t) + (1 - \alpha_l) \cdot \Pr(s_3|h_t)} \right) \cdot \left(\frac{p_t}{1 - p_t} \right) > 1, \tag{B-30}$$

for $0 < \alpha_L < \alpha_H < 1$ and rearranging yields

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1 - p_t)}, \tag{B-31}$$

$$A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)}, \tag{B-32}$$

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\big(p_t + \delta_t^s \cdot (1 - p_t)\big) \cdot \big(p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)\big)}. \tag{B-33}$$

Finally, partial derivatives of δ_t^b and δ_t^s w.r.t. α_L and α_H (i.e., $\frac{\partial \delta_t^b}{\partial \alpha_L} > 0$, $\frac{\partial \delta_t^b}{\partial \alpha_H} > 0$, $\frac{\partial \delta_t^s}{\partial \alpha_L} > 0$ and $\frac{\partial \delta_t^s}{\partial \alpha_L} > 0$) complete the proof.

■

Proof of Lemma 6. Rearranging Eqs. (B-31) and (B-32) obtains

$$\frac{B_{\alpha,t}}{1 - B_{\alpha,t}} = (\delta_t^s)^{-1} \cdot \frac{p_t}{1 - p_t}, \tag{B-34}$$

$$\frac{A_{\alpha,t}}{1 - A_{\alpha,t}} = \delta_t^b \cdot \frac{p_t}{1 - p_t}. \tag{B-35}$$

Analogous to the benchmark model, in the presence of composition uncertainty, the current belief about the payoff is the last transaction price leading to

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot (\delta_t^b), \quad \text{if} \quad D_t = +1, \tag{B-36}$$

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot (\delta_t^s)^{-1}, \quad \text{if} \quad D_t = -1. \tag{B-37}$$

■

Proof of Proposition 7. Given the trading history h_t with b_t buy and s_t number of sell orders at time t, the market maker's belief about the payoff p_t and the proportion of high informed trading π_t respectively follow from Bayes theorem as

$$\begin{aligned} p_t =& \Pr(s_1|h_t) + \Pr(s_3|h_t) \\ =& \Big(p_1 \cdot \pi_1 \cdot \big(1+\alpha_H\big)^{b_t} \cdot \big(1-\alpha_H\big)^{s_t} + p_1 \cdot (1-\pi_1) \cdot \big(1+\alpha_L\big)^{b_t} \cdot \big(1-\alpha_L\big)^{s_t}\Big) \cdot \\ & \Big(p_1 \cdot \pi_1 \cdot (1+\alpha_H)^{b_t} \cdot (1-\alpha_H)^{s_t} + (1-p_1) \cdot \pi_1 \cdot (1-\alpha_H)^{b_t} \cdot (1+\alpha_H)^{s_t} \\ & + p_1 \cdot (1-\pi_1) \cdot (1+\alpha_L)^{b_t} \cdot (1-\alpha_L)^{s_t} + (1-p_1) \cdot (1-\pi_1) \cdot (1-\alpha_L)^{b_t} \cdot (1+\alpha_L)^{s_t}\Big)^{-1}, \end{aligned} \tag{B-38}$$

$$\begin{aligned} \pi_t =& \Pr(s_1|h_t) + \Pr(s_2|h_t) \\ =& \Big(p_1 \cdot \pi_1 \cdot \big(1+\alpha_H\big)^{b_t} \cdot \big(1-\alpha_H\big)^{s_t} + (1-p_1) \cdot \pi_1 \cdot \big(1-\alpha_H\big)^{b_t} \cdot \big(1+\alpha_H\big)^{s_t}\Big) \cdot \\ & \Big(p_1 \cdot \pi_1 \cdot (1+\alpha_H)^{b_t} \cdot (1-\alpha_H)^{s_t} + (1-p_1) \cdot \pi_1 \cdot (1-\alpha_H)^{b_t} \cdot (1+\alpha_H)^{s_t} \\ & + p_1 \cdot (1-\pi_1) \cdot (1+\alpha_L)^{b_t} \cdot (1-\alpha_L)^{s_t} + (1-p_1) \cdot (1-\pi_1) \cdot (1-\alpha_L)^{b_t} \cdot (1+\alpha_L)^{s_t}\Big)^{-1}, \end{aligned} \tag{B-39}$$

where $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$.

(i) Substituting $b_t = s_t$ into Eq. (B-38) obtains $p_t = p_1$ and taking partial derivative of Eq. (B-38) w.r.t. N_t after inserting $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$ obtains $\frac{\partial p_t}{N_t} > 0$.

(ii) Similarly, inserting $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$ into Eq. (B-39) and taking partial derivative w.r.t. N_t obtains $\frac{\partial \pi_t}{N_t} > 0$ when $N_t > 0$ and $\frac{\partial \pi_t}{N_t} < 0$ when $N_t < 0$, leading to $\frac{\partial \pi_t}{|N_t|} > 0$.

■

Proof of Corollary 8. (*i*) Substituting $b_t = s_t$ into Eq. (B-39) obtains

$$\pi_t = \left(\frac{\pi_1 \cdot (1-\alpha_H)^{b_t} \cdot (1+\alpha_H)^{b_t}}{\pi_1 \cdot (1-\alpha_H)^{b_t} \cdot (1+\alpha_H)^{b_t} + (1-\pi_1) \cdot (1-\alpha_L)^{b_t} \cdot (1+\alpha_L)^{b_t}} \right) \cdot \pi_1. \tag{B-40}$$

It follows from $(1-\alpha_H)^{b_t} \cdot (1+\alpha_H)^{b_t} < (1-\alpha_L)^{b_t} \cdot (1+\alpha_L)^{b_t}$ that the first term in Eq. (B-40) is less than 1, leading to $\pi_t < \pi_1$.

(*ii*) Substituting $b_t = t-1$ and $s_t = 0$ into Eq. (B-39) obtains

$$\begin{aligned} \pi_t =& \pi_1 \cdot \big(p_1 \cdot (1+\alpha_H)^{t-1} + (1-p_1) \cdot (1-\alpha_H)^{t-1}\big) \cdot \\ & \Big(\pi_1 \cdot \big(p_1 \cdot (1+\alpha_H)^{t-1} + (1-p_1) \cdot (1-\alpha_H)^{t-1}\big) \\ & + (1-\pi_1) \cdot \big(p_1 \cdot (1+\alpha_L)^{t-1} + (1-p_1) \cdot (1-\alpha_L)^{t-1}\big)\Big)^{-1}. \end{aligned} \tag{B-41}$$

Then $\pi_t > \pi_1$ follows from the fact that $p_1 \cdot (1+\alpha)^{t-1} + (1-p_1) \cdot (1-\alpha)^{t-1}$ is increasing in α if $p_1 \geq 0.5$. The proof of the sequence of sell orders follows similarly.

■

Proof of Corollary 9. *(i)* For the sequence of buy orders, iterating from the first buy order at $t = 1$, Eq. (B-36) leads to

$$\frac{p_t}{1-p_t} = \left(\frac{p_1}{1-p_1}\right) \cdot (\delta_1^b)^{D_1} \cdot \ldots \cdot (\delta_{t-1}^b)^{D_{t-1}} = \left(\frac{p_1}{1-p_1}\right) \cdot \prod_{\tau=1}^{\tau=t-1} \delta_\tau^b \tag{B-42}$$

Substituting $p_1 = 0.5$ into Eq. (B-42) and solving for p_t obtains

$$p_t = \frac{\prod_{i=1}^{t-1} \delta_i^b}{1 + \prod_{i=1}^{t-1} \delta_i^b}. \tag{B-43}$$

Let $\bar{\delta}_t^b$ denote the geometric mean of the informativeness of the buy sequence up to time t. It follows from $N_t = t-1$ that for the sequence of buy orders,

$$\prod_{\tau=1}^{\tau=t-1} \delta_\tau^b = \left(\Big(\prod_{\tau=1}^{\tau=t-1} \delta_\tau^b \Big)^{\frac{1}{t-1}} \right)^{N_t} = (\bar{\delta}_t^b)^{N_t}, \tag{B-44}$$

leading to

$$p_t = \frac{(\bar{\delta}_t^b)^{N_t}}{1 + (\bar{\delta}_t^b)^{N_t}}, \tag{B-45}$$

similar to the benchmark model. *(ii)* The proof of the sell sequence follows similarly.

■

Proof of Proposition 10. *(i)* Given that initially it is equally likely that $\hat{V}$ is high or low (i.e., $p_1 = 0.5$), S_1 is given by

$$S_1 = \frac{\delta_1 - 1}{\delta_1 + 1}, \tag{B-46}$$

where

$$\delta_1 = \delta_1^s = \delta_1^b = \frac{1 + (\pi_1 \cdot \alpha_H + (1-\pi_1) \cdot \alpha_L)}{1 - (\pi_1 \cdot \alpha_H + (1-\pi_1) \cdot \alpha_L)} \tag{B-47}$$

is the initial informativeness of orders. Combining the spread in the presence of composition uncertainty at time t in Eq. (B-33) and the initial spread in Eq. (B-46) obtains

$$S_{\alpha,t} = S_1 + \frac{p_t \cdot (1-p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1}) \cdot (\delta_1 + 1) - \big(p_t + \delta_t^s \cdot (1-p_t)\big) \cdot \big(p_t + (\delta_t^b)^{-1} \cdot (1-p_t)\big) \cdot (\delta_1 - 1)}{\big(p_t + \delta_t^s \cdot (1-p_t)\big) \cdot \big(p_t + (\delta_t^b)^{-1} \cdot (1-p_t)\big) \cdot (\delta_1 + 1)}, \tag{B-48}$$

where the second term in Eq. (B-48) shows the net liquidity distortion ΔS_t relative to the initial spread and is stabilizing when $\Delta S_t < 0$ and destabilizing when $\Delta S_t > 0$.

(ii) The market maker's perceived informativeness of orders following balanced order flow (i.e., $b_t = s_t$) follows from inserting the conditional probabilities of states into Eqs. (B-29) and (B-30) as

$$\delta_t^s = \delta_t^b = \frac{\pi_1 \cdot (1+\alpha_H)^{s_t+1} \cdot (1-\alpha_H)^{s_t} + (1-\pi_t) \cdot (1+\alpha_L)^{s_t+1} \cdot (1-\alpha_L)^{s_t}}{\pi_1 \cdot (1+\alpha_H)^{s_t} \cdot (1-\alpha_H)^{s_t+1} + (1-\pi_1) \cdot (1+\alpha_L)^{s_t} \cdot (1-\alpha_L)^{s_t+1}}. \tag{B-49}$$

Since $p_t = p_1$ for balanced order flow, ΔS_t in Eq. (B-48) reduces to

$$\Delta S_t = \frac{(\delta_t^s - (\delta_t^s)^{-1}) \cdot (\delta_1 + 1) - (1 + \delta_t^s) \cdot (1 + (\delta_t^s)^{-1}) \cdot (\delta_1 - 1)}{(1 + \delta_t^s) \cdot (1 + (\delta_t^s)^{-1}) \cdot (\delta_1 + 1)}, \tag{B-50}$$

which takes a negative value if and only if $\delta_t^s < \delta_1$. Since δ_t^s given in Eq. (B-49) is decreasing in s_t, $\frac{\partial \delta_t^s}{\partial s_t} < 0$, it follows that $\delta_t^s < \delta_1$ is always satisfied, meaning that balanced order flows always stabilize the market.

(iii) For the sequence of sell orders, substituting

$$p_t = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1}}{1 + \prod_{i=1}^{t-1} (\delta_i^s)^{-1}}. \tag{B-51}$$

into the net liquidity distortion ΔS_t in Eq. (B-48) obtains

$$\Delta S_t = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1} \cdot \big(\delta_t^s - (\delta_t^b)^{-1}\big) \cdot (\delta_1 + 1) - \big(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \delta_t^s\big) \cdot \big(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + (\delta_t^b)^{-1}\big) \cdot (\delta_1 - 1)}{\big(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \delta_t^s\big) \cdot \big(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + (\delta_t^b)^{-1}\big) \cdot (\delta_1 + 1)}, \tag{B-52}$$

which takes a positive value if and only if

$$\delta_1 < \frac{2 \cdot \delta_t^s + \prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \prod_{i=1}^{t-1} \delta_i^s \cdot \delta_t^s \cdot (\delta_t^b)^{-1}}{2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \prod_{i=1}^{t-1} \delta_i^s \cdot \delta_t^s \cdot (\delta_t^b)^{-1}}. \tag{B-53}$$

(iv) The proof for the sequence of buy orders follows similarly.

■

Proof of Lemma 11. Inserting the independence conditions of the myopic market maker (i.e., $\Pr(s_1|h_t) = \pi_t^m \cdot p_t^m$, $\Pr(s_2|h_t) = \pi_t^m \cdot (1 - p_t^m)$, $\Pr(s_3|h_t) = (1 - \pi_t^m) \cdot p_t^m$ and $\Pr(s_4|h_t) = (1 - \pi_t^m) \cdot (1 - p_t^m))$ into δ_t^b and δ_t^s in Eqs. (B-29) and (B-30) obtains

$$\delta_t^m = \frac{1 + (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}{1 - (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}, \tag{B-54}$$

which yields

$$B_{\alpha,t}^m = \frac{p_t^m}{p_t^m + \delta_t^m \cdot (1 - p_t^m)}, \tag{B-55}$$

and

$$A_{\alpha,t}^m = \frac{p_t^m}{p_t^m + (\delta_t^m)^{-1} \cdot (1 - p_t^m)}. \tag{B-56}$$

Therefore, as in the benchmark model

$$\frac{p_{t+1}^m}{1 - p_{t+1}^m} = \frac{p_t^m}{1 - p_t^m} \cdot (\delta_t^m)^{D_t}. \tag{B-57}$$

Iterating from the first trade $t = 1$ up to time t obtains

$$\frac{p_t^m}{1 - p_t^m} = \left(\frac{p_1}{1 - p_1}\right) \cdot (\delta_1^m)^{D_1} \cdot \ldots \cdot (\delta_{t-1}^m)^{D_{t-1}} = \left(\frac{p_1}{1 - p_1}\right) \cdot \prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau} = \prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau}, \tag{B-58}$$

leading to

$$p_t^m = \frac{\prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau}}{1 + \prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau}}. \tag{B-59}$$

We now show that $\prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau}$ is given by $\bar{\delta}_t^{\bar{N}_t}$. Let

$$\ln \bar{\delta}_t^m = \frac{1}{t-1} \cdot \sum_{i=1}^{t-1} \ln \delta_i^m = \frac{1}{t-1} \cdot \ln \prod_{i=1}^{t-1} \delta_i^m = \ln \left(\prod_{i=1}^{t-1} \delta_i^m \right)^{\frac{1}{t-1}}, \tag{B-60}$$

leading to $\bar{\delta}_t^m = \left(\prod_{i=1}^{t-1} \delta_i^m \right)^{\frac{1}{t-1}}$. Taking the log of $\prod_{\tau=1}^{t-1} (\delta_\tau^m)^{D_\tau}$, and multiplying and dividing with $\sum_{i=1}^{t-1} \ln \delta_i^m$ obtains

$$\begin{aligned}\ln\left(\prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}\right) &= \sum_{\tau=1}^{t-1} D_\tau \cdot \ln \delta_\tau^m = \sum_{i=1}^{t-1} \ln \delta_i^m \cdot \sum_{\tau=1}^{t-1} D_\tau \cdot \frac{\ln \delta_\tau^m}{\sum_{i=1}^{t-1} \ln \delta_i^m} \\ &= \ln \bar{\delta}_t^m \cdot \sum_{\tau=1}^{t-1} D_\tau \cdot \frac{(t-1) \cdot \ln \delta_\tau^m}{\sum_{i=1}^{t-1} \ln \delta_i^m} = \ln \bar{\delta}_t^m \cdot \sum_{\tau=1}^{t-1} D_\tau \cdot w_\tau = \ln \bar{\delta}_t^m \cdot \bar{N}_t,\end{aligned} \tag{B-61}$$

where $w_\tau = \frac{(t-1)\cdot \ln \delta_\tau^m}{\sum_{i=1}^{t-1} \ln \delta_i^m}$ and $\bar{N}_t = \sum_{\tau=1}^{t-1} D_\tau \cdot w_\tau$. Hence, $\prod_{\tau=1}^{\tau=t-1} (\delta_\tau^m)^{D_\tau} = \bar{\delta}_t^{\bar{N}_t}$.

Lastly, the myopic market maker's learning about the proportion of informed traders after a buy and a sell follows from Bayes' theorem respectively as

$$\begin{aligned}\pi_{t+1}^m = \Pr\{\hat{\alpha} = \alpha_H | h_t, D_t = +1\} &= \frac{\Pr\{D_t = +1 | h_t, \hat{\alpha} = \alpha_H\} \cdot \Pr\{\hat{\alpha} = \alpha_H | h_t\}}{\Pr\{D_t = +1 | h_t\}} \\ &= \frac{1 + \alpha_H \cdot (2 \cdot p_t^m - 1) \cdot (2 \cdot q - 1)}{1 + \left(\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L\right) \cdot (2 \cdot p_t^m - 1) \cdot (2 \cdot q - 1)} \cdot \pi_t^m,\end{aligned} \tag{B-62}$$

$$\begin{aligned}\pi_{t+1}^m = \Pr\{\hat{\alpha} = \alpha_H | h_t, D_t = -1\} &= \frac{\Pr\{D_t = -1 | h_t, \hat{\alpha} = \alpha_H\} \cdot \Pr\{\hat{\alpha} = \alpha_H | h_t\}}{\Pr\{D_t = -1 | h_t\}} \\ &= \frac{1 + \alpha_H \cdot (1 - 2 \cdot p_t^m) \cdot (2 \cdot q - 1)}{1 + \left(\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L\right) \cdot (1 - 2 \cdot p_t^m) \cdot (2 \cdot q - 1)} \cdot \pi_t^m,\end{aligned} \tag{B-63}$$

which combined leads to Eq. (47).

■

REFERENCES

Anand, A. and Venkataraman, K. (2016), 'Market conditions, fragility, and the economics of market making', *Journal of Financial Economics* **121**(2), 327–349.

Avery, C. and Zemsky, P. (1998), 'Multidimensional uncertainty and herd behavior in financial markets', *American Economic Review* **88**(4), 724–748.

Banerjee, S. and Green, B. (2015), 'Signal or noise? uncertainty and learning about whether other traders are informed', *Journal of Financial Economics* **117**(2), 398–423.

Barlevy, G. and Veronesi, P. (2003), 'Rational panics and stock market crashes', *Journal of Economic Theory* **110**(2), 234–263.

Blume, L., Easley, D. and O'Hara, M. (1994), 'Market statistics and technical analysis: The role of volume', *Journal of Finance* **49**(1), 153–181.

CFTC-SEC (2010a), 'Preliminary findings regarding the market events of May 6, 2010', *Staff Report* .

CFTC-SEC (2010b), 'Findings regarding the market events of May 6, 2010', *Staff Report* .

Chiyachantana, C., Jain, P. K., Jiang, C. and Sharma, V. (2017), 'Permanent price impact asymmetry of trades with institutional constraints', *Journal of Financial Markets* **36**, 1–16.

Chordia, T., Roll, R. and Subrahmanyam, A. (2002), 'Order imbalance, liquidity, and market returns', *Journal of Financial Economics* **65**(1), 111–130.

Easley, D., Kiefer, N. M. and O'Hara, M. (1997), 'The information content of the trading process', *Journal of Empirical Finance* **4**(2), 159–186.

Easley, D., López de Prado, M. M. and O'Hara, M. (2011), 'The microstructure of the flash crash: flow toxicity, liquidity crashes, and the probability of informed trading', *Journal of Portfolio Management* **37**(2), 118–128.

Easley, D., López de Prado, M. M. and O'Hara, M. (2012), 'Flow toxicity and liquidity in a high-frequency world', *Review of Financial Studies* **25**(5), 1457–1493.

Easley, D. and O'Hara, M. (1992), 'Time and the process of security price adjustment', *Journal of Finance* **47**(2), 577–605.

Easley, D. and O'Hara, M. (2010), 'Liquidity and valuation in an uncertain world', *Journal of Financial Economics* **97**(1), 1–11.

Engle, R. F. and Patton, A. J. (2004), 'Impacts of trades in an error-correction model of quote prices', *Journal of Financial Markets* **7**(1), 1–25.

Gao, F., Song, F. and Wang, J. (2013), 'Rational expectations equilibrium with uncertain proportion of informed traders', *Journal of Financial Markets* **16**(3), 387–413.

Gennotte, G. and Leland, H. (1990), 'Market liquidity, hedging, and crashes', *American Economic Review* **80**(5), 999–1021.

Gervais, S. (1997), 'Essays on market microstructure with uncertain information precision, optimal information sales, and overconfidence.', *Ph.D. dissertation, University of California, Berkeley* .

Glosten, L. R. and Milgrom, P. R. (1985), 'Bid, ask and transaction prices in a specialist market with heterogeneously informed traders', *Journal of Financial Economics* **14**(1), 71–100.

Glosten, L. R. and Putnins, T. J. (2016), 'Welfare costs of informed trade', *Working Paper, Columbia Business School and UTS Business School* .

Hasbrouck, J. (1991), 'Measuring the information content of stock trades', *Journal of Finance* **46**(1), 179–207.

Hausman, J. A., Lo, A. W. and MacKinlay, A. C. (1992), 'An ordered probit analysis of transaction stock prices', *Journal of Financial Economics* **31**(3), 319–379.

Hong, H. and Stein, J. C. (2003), 'Differences of opinion, short-sales constraints, and market crashes', *Review of Financial Studies* **16**(2), 487–525.

Keim, D. B. and Madhavan, A. (1996), 'The upstairs market for large-block transactions: Analysis and measurement of price effects', *Review of Financial Studies* **9**(1), 1–36.

Kirilenko, A. A., Kyle, A. S., Samadi, M. and Tuzun, T. (2017), 'The flash crash: High frequency trading in an electronic market', *Journal of Finance* **72**(3), 967–998.

Kraus, A. and Stoll, H. R. (1972), 'Price impacts of block trading on the New York Stock Exchange', *Journal of Finance* **27**(3), 569–588.

Kyle, A. S. (1985), 'Continuous auctions and insider trading', *Econometrica* **15**(35), 1315–1335.

Romer, D. (1993), 'Rational asset-price movements without news', *American Economic Review* **83**(5), 1112–30.

Saar, G. (2001), 'Price impact asymmetry of block trades: An institutional trading explanation', *Review of Financial Studies* **14**(4), 1153–1181.

Scholes, M. S. (2000), 'Crisis and risk management', *American Economic Review* **90**(2), 17–21.

U.S. Dept. of the Treasury, Fed. Reserve System, Fed. Reserve Bank of New York, SEC and CFTC (2014), 'The U.S. Treasury Market on October 15, 2014', *Staff Report* .